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On the harmonic superspace geometry of $(4,4)$ supersymmetric sigma models with torsion

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Abstract

Starting from the dual action of $(4,4)$ $2D$ twisted multiplets in the harmonic superspace with two independent sets of $SU(2)$ harmonic variables, we present its generalization which hopefully provides an off-shell description of general $(4,4)$ supersymmetric sigma models with torsion. Like the action of the torsionless $(4,4)$ hyper-Kähler sigma models in the standard harmonic superspace, it is characterized by a number of superfield potentials. They depend on n copies of a triple of analytic harmonic $(4,4)$ superfields. As distinct from the hyper-Kähler case, the potentials prove to be severely constrained by the self-consistency condition which stems from the commutativity of the left and right harmonic derivatives. We show that for $n = 1$ these constraints reduce the general action to that of $(4,4)$ twisted multiplet, while for $n \geq 2$ there exists a wide class of new actions which cannot be written only via twisted multiplets. Their most striking feature is the nonabelian and in general nonlinear gauge invariance which substitutes the abelian gauge symmetry of the dual action of twisted multiplets and ensures the correct number of physical degrees of freedom. We show, on a simple example, that these actions describe sigma models with non-commuting left and right complex structures on the bosonic target.

1 Introduction

An interesting and important class of two-dimensional supersymmetric sigma models consists of those with $(4,4)$ worldsheet supersymmetry. The main reason of current interest to them is that they can provide non-trivial backgrounds for $d=4$ strings (see, e.g., [1]). Relevant bosonic target manifolds in general possess a nontrivial torsion and two triplets of covariantly constant complex structures (left and right ones) which may be mutually commuting or non-commuting [2, 3]. The $(4,4)$ sigma models which can be obtained via a direct dimensional reduction of $N=2$ $4D$ sigma models constitute merely a subclass in the general variety of $(4,4)$ $2D$ sigma models; their bosonic target manifolds are hyper-Kähler (or quaternionic-Kähler in the case of local supersymmetry) and so are torsionless and possess only one set of complex structures [4]. A manifestly supersymmetric off-shell description of this latter type of sigma models has been given in [5 - 8] in the harmonic $N=2$ $4D$ (or $(4,4)$ $2D$) superspace with one set of harmonic variables parametrizing the $SU(2)$ automorphism group of $N=2$ $4D$ ($(4,4)$ $2D$) supersymmetry [9, 10]. Later on, an analogous formulation with the use of the same type of harmonic superspace has been developed for sigma models with heterotic worldsheet $(4,0)$ supersymmetry [11] (in these models bosonic target manifolds in general possess a torsion). One of the basic advantages of such off-shell formulations is that they visualize the intrinsic geometric features of the relevant target manifolds: the corresponding superfield Lagrangians turn out to coincide with (or to be directly related to) the unconstrained potentials underlying the target geometries, while the involved superfields are identified with coordinates of some important subspaces of the target manifolds, the analytic subspaces. For instance, in the torsionless $(4,4)$ case the harmonic superfield Lagrangian is recognized as the hyper-Kähler potential [7,8]. Unconstrained off-shell formulations provide us with an efficient tool for the explicit computation of the bosonic metrics (e.g., hyper-Kähler ones in the torsionless $(4,4)$ case) which automatically satisfy all the restrictions placed by extended supersymmetry [8, 12]. Note that these restrictions in their own right [2,3] give no any explicit recipe for calculating the metrics.

Since the full automorphism group of $(4,4)$ supersymmetry in two dimensions is $SO(4)_L \times SO(4)_R$, there arises a possibility to consider more general types of harmonic superspaces compared to the one utilized in [5 - 12]. In [13] A. Sutulin and the author have constructed the $(4,4)$ $2D$ harmonic superspace which involves two independent sets of harmonic variables parametrizing two commuting $SU(2)$ automorphism groups in the left and right light-cone sectors[1], $SU(2)_L$ and $SU(2)_R$ (the automorphism $SU(2)$ group of the conventional $(4,4)$ $2D$ harmonic superspace is a diagonal in the product $SU(2)_L \times SU(2)_R$). We have shown how to describe in this $SU(2)\times SU(2)$ harmonic superspace the $(4,4)$ twisted supermultiplet [2, 15] and presented the most general off-shell action of the latter as an integral over an analytic subspace of this superspace. The action involves the standard number of auxilary fields (four bosonic ones) and, in accord with arguments of Refs. [2, 16], corresponds to a general $(4,4)$ supersymmetric sigma model with torsion and mutually commuting sets of left and right complex structures. A new dual form of the action in terms of unconstrained analytic superfields with an infinite number of auxiliary fields has been also given. An interesting peculiarity of the dual action is

[1] See also [14].

the abelian gauge invariance which ensures the on-shell equivalence of this action to the original one. We argued that this form of the action is a good starting point to attack the problem (as yet unsolved) of constructing a manifestly $(4,4)$ supersymmetric off-shell description of $(4,4)$ sigma models with non-commuting left and right complex structures. These models *cannot* be described only in terms of $(4,4)$ twisted multiplets [2, 16], so one is led to seek for such generalizations of the dual action which would not allow an equivalent formulation via $(4,4)$ harmonic superfields representing twisted multiplets [2].

In the present paper we generalize the dual action of $(4,4)$ twisted multiplet along these lines. As the main result, we find a wide class of off-shell $(4,4)$ sigma model actions with a nonabelian generalization of the abelian gauge invariance of the dual action. They cannot be written only through $(4,4)$ twisted superfields and, for this reason, can be thought of as corresponding to the aforementioned more general type of $(4,4)$ sigma models. We explicitly demonstrate the non-commutativity of the left and right complex structures for some interesting particular type of the actions constructed.

Our consideration is largely based upon an analogy with the description of torsionless $N=2$ $4D$ ($(4,4)$ in two dimensions) hyper-Kähler supersymmetric sigma models in the standard (having one set of harmonic variables) harmonic superspace. So we start in Sect.2 by recapitulating salient features of this description. Then in Sect.3 we recollect the basic facts about the $SU(2)\times SU(2)$ harmonic superspace and off-shell description of the twisted $(4,4)$ multiplet in its framework. In Sect.4 we discuss the dual action of the latter which involves n copies of a triple of unconstrained analytic superfields, and construct its most general extension, proceeding from the analogy with the general hyper-Kähler $(4,4)$ sigma model. This extension includes a few superfield potentials which, as distinct from the unconstrained potentials of the hyper-Kähler $(4,4)$ action, prove to be severely constrained by the integrability condition coming from the commutativity of the left and right harmonic derivatives. In Sect.5 we elaborate the $n=1$ example (with four-dimensional bosonic target) and show that the integrability constraint just mentioned reduces the general $n=1$ action to that of one twisted multiplet. In Sect.6 we return to considering the generic $n\geq 2$ action. We partially solve the integrability constraint and find a wide variety of the actions which in general do not admit a presentation through the twisted $(4,4)$ superfields and so encompass sigma models with commuting as well as non-commuting left and right complex structures. Besides the inevitable presence of an infinite number of auxiliary fields, one more intriguing feature of these actions is the nonabelian and in general nonlinear gauge invariance which generalizes the abelian gauge symmetry of the dual action of twisted multiplets and is necessary for ensuring the correct number of physical fields ($4n$ bosonic and $8n$ fermionic ones). Surprisingly, the actions constructed are bi-harmonic analogs of the action of the so called Poisson gauge theory [20] which is a nonlinear extension of Yang-Mills theory. We discuss in some detail their interesting subclass, viz. direct bi-harmonic analogs of the two-dimensional Yang-Mills action. We compute, to the first order in fields, the relevant bosonic metric and torsion potential and show that the left and right complex structures on the bosonic target *do not commute.*

[2]For other proposals of how to describe off shell $(4,4)$ and $(2,2)$ sigma models with non-commuting complex structures see Refs. [14, 17, 18].

2 Sketch of $(4,4)$ sigma models in standard harmonic superspace

To make further consideration self-contained, it is instructive to start with a brief review of the off-shell formulation of $(4,4)$ sigma models in $(4,4)$ $2D$ harmonic superspace which is obtained by dimensional reduction from the standard $N=2$ $4D$ harmonic superspace [9, 10]. The relevant action contains no torsion in the bosonic part; the bosonic target space metric is necessarily hyper-Kähler [4].

The sigma models in question are described in terms of unconstrained analytic harmonic superfields $q^{(+)\,M}(\zeta,u)$ $(M=1,2,...2n)$ defined on the $(2|4)$ dimensional $(4,4)$ $2D$ analytic harmonic superspace (see [5, 6] for details and terminology).

$$(\zeta,u)=(z^{++},z^{--},\theta^{(+)+},\bar{\theta}^{(+)+},\theta^{(+)-},\bar{\theta}^{(+)-},u^{(+)}_i,u^{(-)}_j)\ . \tag{2.1}$$

Here, the harmonic variables $u^{(\pm)i}$,

$$u^{(+)i}u^{(-)}_i=1\ ,$$

parametrize the two-sphere $S^2\sim SU(2)/U(1)$, $SU(2)$ being the diagonal subgroup in the product of two independent $SU(2)$ automorpism groups (the left and right ones) of the $(4,4)$ $2D$ Poincaré superalgebra. The indices $\pm$ in the parentheses refer to the harmonic $U(1)$ charge, other $\pm$'s are $2D$ light-cone indices.

The general action of superfields $q^{(+)\,M}$ yields in the bosonic sector a generic sigma model on $4n$ dimensional hyper-Kähler manifold. The action is given by the following integral over the $(4,4)$ $2D$ analytic harmonic superspace

$$S_q=\int\mu^{(-4)}\{L^{(+)}_M(q^{(+)},u)D^{(+2)}q^{(+)\,M}+L^{(+4)}(q^{(+)},u)\}\ . \tag{2.2}$$

The object

$$D^{(+2)}=\partial^{(+2)}+2i(\theta^{(+)+}\bar{\theta}^{(+)+}\partial_{++}+\theta^{(+)-}\bar{\theta}^{(+)-}\partial_{--})\ ,\ \ (\partial^{(+2)}=u^{(+)i}\frac{\partial}{\partial u^{(-)i}})\ , \tag{2.3}$$

is the analyticity-preserving harmonic derivative, $\mu^{(-4)}$ is the analytic superspace integration measure

$$\mu^{(-4)}=d^6\zeta\ [du]=d^2zd^2\theta^{(+)+}d^2\theta^{(+)-}\ [du].$$

Two arbitrary potentials in the superfield Lagrangian in (2.2), $L^{(+)}_M(q^{(+)\,M},u)$ and $L^{(+4)}(q^{(+)\,M},u)$, encode (locally) all the information about the relevant bosonic hyper-Kähler manifold. The fields parametrizing the latter appear as the first components in the harmonic and θ expansions of $q^{(+)\,M}$

$$q^{(+)\,M}(\zeta,u)=q^{iM}(z)u^{(+)}_i+...\ .$$

The needed number of independent real fields in $q^{iM}(z)$ (just $4n$) comes out as a result of imposing the reality condition on the superfields $q^{(+)\,M}$: $(\widetilde{q^{(+)\,M}})=\Omega_{MN}q^{+\,N}$, where Ω_{MN}

is a constant skew-symmetric matrix and the generalized involution "$\sim$" was defined in Refs. [9, 10].

The quantities $L^{(+)}_M$ and $L^{(+4)}$ have a clear geometric meaning: these are the hyper-Kähler potentials, the basic objects of unconstrained formulation of hyper-Kähler geometry given for the first time in [7]. There we started from the standard definition of this geometry as a Riemann geometry with restricted holonomy group (in case of $4n$ dimensional manifold it should be a subgroup of $Sp(n)$). We extended the original (arbitrary) hyper-Kähler manifold by a set of harmonic variables which parametrize the $SU(2)$ group rotating complex structures and then solved the constraints on the curvature by passing to a new, analytic basis in such a harmonic extension. The main feature of this extension which is visualized by passing to the analytic basis is the existence of an analytic subspace with twice as few coordinates compared to the manifold one started with (besides the harmonic variables the number of which is the same). The basic geometric objects which solve the hyper-Kähler constraints are just $L^{(+)}_M$ and $L^{(+4)}$ living as unconstrained functions on this analytic subspace.

The fact that the action of most general $N=2$ $4D$ ($(4,4)$ upon the reduction to two dimensions) supersymmetric sigma model is expressed via $L^{(+)}_M$ and $L^{(+4)}$, while identifying superfields $q^{(+)\,M}$ with coordinates of an analytic subspace of the harmonic extension of the target hyper-Kähler manifold, and the automorphism $SU(2)$ with the $SU(2)$ group rotating complex structures on this manifold, makes manifest the remarkable one-to-one correspondence between $N=2$ $4D$ ($(4,4)$ $2D$) supersymmetry and hyper-Kähler geometry [4]. There exists a clear analogy with $N=1$ $4D$ ($(2,2)$ $2D$) sigma models: the most general off-shell superfield Lagrangian of the latter can be interpreted as some Kähler potential, with the involved chiral superfields as the coordinates of the associated Kähler manifold. This makes manifest the one-to-one correspondence between Kähler geometry and $N=1$ $4D$ ($(2,2)$ $2D$) supersymmetry [19].

It is important to point out that the superfield action (2.2) has been written and interpreted as the most general $N=2$ $4D$ supersymmetric sigma model action [5, 6] *prior* to recognizing the potentials $L^{(+)}_M$ and $L^{(+4)}$ as the basic objects of hyper-Kähler geometry and deducing them from the primary principles of the latter in [7]. Many characteristic features of the analytic space formulation of this geometry can be read off by inspecting the action (2.2). For instance, it is invariant under arbitrary analytic reparametrizations of $q^{(+)\,M}$

$$\delta q^{(+)\,M} = \Lambda^{(+)\,M}(q^{(+)},u)\ , \tag{2.4}$$

provided that $L^{(+)}_M$ and $L^{(+4)}$ transform as

$$\delta L^{(+)}_M = -L^{(+)}_N \frac{\partial \Lambda^{(+)\,N}}{\partial q^{(+)\,M}}\ ,\ \ \delta L^{(+4)} = -L^{(+)}_N \partial^{(+2)} \Lambda^{(+)\,N}\ , \tag{2.5}$$

as well as under the following transformations called in [6] the hyper-Kähler ones (because these are a direct analog of Kähler transformations $K(x,\bar{x}) \Rightarrow K(x,\bar{x}) + \Lambda(x) + \bar{\Lambda}(\bar{x})$)

$$\delta q^{(+)\,M} = 0\ ,\ \delta L^{(+)}_M = \frac{\partial \Lambda^{(+2)}}{\partial q^{(+)\,M}}\ ,\ \delta L^{(+4)} = \partial^{(+2)}\Lambda^{(+2)}\ ,\ \ \Lambda^{(+2)} = \Lambda^{(+2)}(q^{(+)},u)\ . \tag{2.6}$$

Here $\partial^{(+2)}$ acts only on the explicit harmonics in the arguments of $\Lambda^{(+)\,M}$, $\Lambda^{(+2)}$. The geometric origin of these transformations have been fully understood later on [7] within the analytic space formulation of hyper-Kähler manifolds. Note that these invariances can be used to gauge $L^{(+)}_M$ into its "flat" part $q^{(+)\,M}$

$$L^{(+)}_M = -\Omega_{MN} q^{(+)\,N} \,, \tag{2.7}$$

thus demonstrating that the only essential hyper-Kähler potential is $L^{(+4)}$ (the sign "$-$" in (2.7) ensures the correct sign of the kinetic term of physical bosonic fields in the component action).

The equation of motion for $q^{(+)\,M}$ following from (2.2)

$$\begin{aligned} D^{(+2)} q^{(+)\,M} &= -H^{MN}\left(\frac{\partial L^{(+4)}}{\partial q^{(+)\,N}} - \partial^{(+2)} L^{(+)}_N\right) \,, \\ H^{MN} H_{NT} &= \delta^M_T \,, \quad H_{NT} = \frac{\partial L^{(+)}_N}{\partial q^{(+)\,T}} - \frac{\partial L^{(+)}_T}{\partial q^{(+)\,N}} \end{aligned} \tag{2.8}$$

also has a nice geometric interpretation. Defining the target space harmonic derivative $\mathcal{D}^{(+2)}$ which acts in the target analytic subspace spanned by the coordinates $q^{(+)\,M}, u^{(\pm)i}$

$$\mathcal{D}^{(+2)} = \partial^{(+2)} + D^{(+2)} q^{(+)\,M} \frac{\partial}{\partial q^{(+)\,M}} \equiv \partial^{(+2)} + E^{(+3)\,M} \frac{\partial}{\partial q^{(+)\,M}} \,, \tag{2.9}$$

one observes that eq. (2.8) is none other than the expression of the target space analytic vielbein $E^{(+3)\,M}$ in terms of the hyper-Kähler potentials [7]. Moreover, in the sigma model context it is precisely eq. (2.8) that tells us that $D^{(+2)} q^{(+)\,M} \equiv E^{(+3)\,M}$ is target-space analytic and, hence, that $\mathcal{D}^{(+2)}$ (2.9) preserves the target space harmonic analyticity.

Summarizing, a manifestly supersymmetric off-shell formulation of general torsionless $(4,4)$ sigma models in terms of unconstrained harmonic-analytic superfields allows one to independently find out the basic elements of the analytic space geometry of the target hyper-Kähler manifolds. So, one way to reveal the intrinsic geometry of the target manifolds of torsionful $(4,4)$ sigma models is to construct the appropriate general off-shell superfield formulation. This will be the subject of the next Sections.

In what follows we will refer to a slightly different representation of the general action (2.2). Let us split the target space world index M as $M = (i\alpha)$, $i = 1,2;\ \alpha = 1,2,...n$ and, using the completeness property of harmonics

$$u^{(+)i} u^{(-)k} - u^{(+)k} u^{(-)i} = \epsilon^{ki} \,, \ \ (\epsilon^{12} = -\epsilon_{12} = -1) \,,$$

equivalently re-express $q^{(+)\,M} = q^{(+)\,i\alpha}$ through the pair of analytic superfields $\omega^\alpha(\zeta, u)$, $l^{(+2)\,\alpha}(\zeta, u)$

$$\begin{aligned} q^{(+)\,i\alpha} &= u^{(+)i}\omega^\alpha - u^{(-)i} l^{(+2)\,\alpha} \,, \\ \omega^\alpha &= u^{(-)}_i q^{(+)\,i\alpha} \,, \quad l^{(+2)\,\alpha} = u^{(+)}_i q^{(+)\,i\alpha} \,. \end{aligned} \tag{2.10}$$

In terms of these superfields the action (2.2) can be rewritten as

$$S_{\omega,l} = \int \mu^{(-4)} \{ L^{(+2)}_\alpha(\omega, l, u) D^{(+2)} \omega^\alpha + L_\alpha(\omega, l, u) D^{(+2)} l^{(+2)\,\alpha} + \tilde{L}^{(+4)}(\omega, l, u) \} \,. \tag{2.11}$$

To know the precise form of the relation between the potentials in (2.11) and the previous ones $L^{(+)}_M$, $L^{(+4)}$, as well as the ω, l realization of groups (2.4), (2.6), is of no need for our further purposes. We only note that the potentials $L^{(+2)}_\alpha$, L_α are also pure gauge. They can be gauged into their flat parts

$$L^{(+2)}_\alpha = l^{(+2)\,\alpha}\ , \quad L_\alpha = -\omega^\alpha\ , \tag{2.12}$$

where, without loss of generality, we have chosen $\Omega_{MN} = \epsilon_{ij}\delta_{\alpha\beta}$.

Note that in this gauge and with $\tilde{L}^{(+4)}$ displaying no dependence on ω^α, the general action reduces to

$$S_l = \int \mu^{(-4)}\{-2\omega^\alpha D^{(+2)} l^{(+2)\,\alpha} + \tilde{L}^{(+4)}(l,u)\}\ . \tag{2.13}$$

This reduced action is on-shell equivalent to the general action of $N = 2$ tensor multiplets. Indeed, varying (2.13) with respect to ω^α, we arrive at the action which contains only the $\tilde{L}^{(+4)}(l,u)$ part,

$$S_l = \int \mu^{(-4)}\tilde{L}^{(+4)}(l,u)\ , \tag{2.14}$$

with the superfield $l^{(+)\,\alpha}$ subjected to the constraint

$$D^{(+2)} l^{(+2)\,\alpha} = 0\ . \tag{2.15}$$

This is just the harmonic superspace action and constraint of $N = 2$ $4D$ ((4,4) $2D$) tensor multiplet [5]. Alternatively, one could vary (2.13) with respect to $l^{(+2)\,\alpha}$ and, expressing $l^{(+2)\,\alpha}$ from the resulting algebraic equation as a function of $D^{(+2)}\omega^\alpha$, rewrite (2.13) through the unconstrained analytic superfields ω^α. This kind of $N = 2$ $4D$ ((4,4) $2D$) duality relates to each other two different off-shell descriptions of the same scalar supermultiplet (4+4 components on shell): with a finite number of auxiliary fields (l representation of the action) and with an infinite number of auxiliary fields (ω representation of the action). Note that the passing to the ω form is possible for the general action (2.11) as well, because for the superfield $l^{(+2)\,\alpha}$ the equation of motion is always algebraic,

$$l^{(+)\,\alpha} \sim D^{(+2)}\omega^\alpha + ...\ , \tag{2.16}$$

and by means of this equation $l^{(+2)\,\alpha}$ can be expressed, at least iteratively, in terms of ω^α. Actually, the l, ω and ω actions are the first and second order forms of the same general (4,4) supersymmetric hyper-Kähler sigma model action.

3 $SU(2) \times SU(2)$ harmonic superspace

In our further notation we will basically follow Ref. [13] with minor deviations. We start with some definitions.

The standard (4,4) $2D$ superspace is defined as

$$\mathbf{S}^{(1,1|4,4)} = (x^{++}, x^{--}, \theta^{+\,i\underline{k}}, \theta^{-\,a\underline{b}}).$$

Here $+,-$ are light-cone indices and $i, \underline{k}, a, \underline{b}$ are doublet indices of four commuting $SU(2)$ groups which constitute the full automorphism group $SO(4)_L \times SO(4)_R$ of $(4,4)$ $2D$ Poincaré superalgebra. The harmonic $(4,4)$ superspace constructed in [13] is an extension of $\mathbf{S}^{(1,1|4,4)}$ by two independent sets of harmonic variables $u^{\pm 1}_i,\ v^{\pm 1}_a$, each parametrizing one of the $SU(2)$ factors of $SO(4)_L$ and $SO(4)_R$, respectively (we denote them by $SU(2)_L$ and $SU(2)_R$):

$$\mathbf{HS}^{(1+2,1+2|4,4)} = \mathbf{S}^{(1,1|4,4)} \otimes (u^{\pm 1}_i, v^{\pm 1}_a)\ , \quad u^{1\,i} u^{-1}_i = 1,\ \ v^{1\,a} v^{-1}_a = 1\ .$$

The harmonics u and v carry two independent $U(1)$ charges which are assumed to be strictly conserved (like in the standard $N=2$ $4D$ harmonic superspace [9],GIKOS). This requirement actually implies u and v to parametrize the 2-spheres $S^2_L \sim SU(2)_L/U(1)_L$ and $S^2_R \sim SU(2)_R/U(1)_R$. All superfields given on $\mathbf{HS}^{(1+2,1+2|4,4)}$ possess two definite $U(1)$ charges and, correspondingly, are assumed to be decomposable in the double harmonic series on the above 2-spheres.

Like in the $N=2$ $4D$ case, the main merit of passing to the $(4,4)$ harmonic superspace in question is the existence of an analytic subspace in it which is closed under $(4,4)$ supersymmetry and includes half of the original odd coordinates

$$\mathbf{AS}^{(1+2,1+2|2,2)} = (z^{++}, z^{--}, \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{a}}, u^{\pm 1}_i,\ v^{\pm 1}_a) \equiv (\zeta^{\mu}, u^{\pm 1}_i,\ v^{\pm 1}_a)\ , \tag{3.1}$$

where

$$\theta^{1,0\,\underline{i}} = \theta^{+\,i\underline{i}}\, u^1_i\ , \theta^{0,1\,\underline{a}} = \theta^{-\,a\underline{a}}\, v^1_a\ ,$$

and the relation between $z^{\pm\pm}$ and $x^{\pm\pm}$ can be found in [13]. Superfields given on the superspace (3.1), $\Phi^{p,q}(\zeta, u, v)$ (p and q are values of the left and right harmonic $U(1)$ charges), are called analytic $(4,4)$ superfields.

The analytic superspace (3.1) is real with respect to the generalized involution "$\sim$" which is the product of ordinary complex conjugation and an antipodal map of the 2-spheres $SU(2)_L/U(1)_L$ and $SU(2)_R/U(1)_R$

$$(\widetilde{\theta^{1,0\,\underline{i}}}) = \theta^{1,0}_{\underline{i}}\ ,\ \ (\widetilde{u^{\pm 1\,i}}) = -u^{\pm 1}_i\ , \tag{3.2}$$

(and similarly for $\theta^{1\,\underline{a}}, v^{\pm 1}_a$). The analytic superfields $\Psi^{p,q}$ can be chosen real with respect to this involution, provided $|p+q| = 2n$

$$(\widetilde{\Psi^{p,q}}) = \Psi^{p,q}\ , |p+q| = 2n\ . \tag{3.3}$$

In what follows we will need the fact of existence of two mutually commuting sets of derivatives with respect to harmonics $u^{\pm 1\,i}$ and $v^{\pm 1\,a}$, each forming an $SU(2)$ algebra

$$\begin{aligned} \partial^{\pm 2,0} &= u^{\pm 1\,i} \frac{\partial}{\partial u^{\mp 1\,i}}\ ,\ \ \partial^0_u \ = \ u^{1\,i} \frac{\partial}{\partial u^{1\,i}} - u^{-1\,i} \frac{\partial}{\partial u^{-1\,i}} \\ \partial^{0,\pm 2} &= v^{\pm 1\,a} \frac{\partial}{\partial v^{\mp 1\,a}}\ ,\ \partial^0_v \ = \ v^{1\,a} \frac{\partial}{\partial v^{1\,a}} - v^{-1\,a} \frac{\partial}{\partial v^{-1\,a}}\ . \end{aligned} \tag{3.4}$$

The full analyticity preserving harmonic derivatives $D^{2,0}$, D^0_u, $D^{0,2}$, D^0_v, when applied on analytic superfields, are given by the expressions

$$\begin{aligned} D^{2,0} &= \partial^{2,0} + i\theta^{1,0}\theta^{1,0}\partial_{++}\ ,\ \ D^{0,2} \ = \ \partial^{0,2} + i\theta^{0,1}\theta^{0,1}\partial_{--} \\ D^0_u &= \partial^0_u + \theta^{1,0\,\underline{i}} \frac{\partial}{\partial \theta^{1,0\,\underline{i}}}\ ,\ \ D^0_v \ = \ \partial^0_v + \theta^{0,1\,\underline{a}} \frac{\partial}{\partial \theta^{0,1\,\underline{a}}}\ . \end{aligned} \tag{3.5}$$

The operators D^0_u, D^0_v count the $U(1)$ charges of analytic $(4,4)$ superfields

$$D^0_u \Phi^{p,q}(\zeta,u,v) = p\Phi^{p,q}(\zeta,u,v)\ , \quad D^0_v \Phi^{p,q}(\zeta,u,v) = q\Phi^{p,q}(\zeta,u,v)\ . \tag{3.6}$$

The last topic of this Section will be the harmonic superspace off-shell description of $(4,4)$ twisted chiral multiplet. Actually, the fact that this important multiplet has a natural formulation in the framework of the $(4,4)$ $SU(2)\times SU(2)$ harmonic superspace furnishes the main motivation in favour of the relevance of the latter to $(4,4)$ sigma models with torsion.

The multiplet in question is represented by an analytic $(4,4)$ superfield $q^{1,1}(\zeta,u,v)$ obeying the harmonic constraints

$$D^{2,0}q^{1,1} = D^{0,2}q^{1,1} = 0\ . \tag{3.7}$$

They leave in $q^{1,1}$ $8+8$ independent components [13], that is precisely the off-shell field content of $(4,4)$ twisted multiplet [2, 15]. Notice a formal similarity of the constraints (3.7) to the constraint (2.15) defining $N=2$ tensor multiplet in the harmonic $N=2$ $4D$ superspace. The crucial difference between either constraints is that (2.15) implies a differential condition for a vector component of the relevant superfield, requiring it to be divergenceless, while this is not the case for the constraints (3.7). These latter constraints are purely algebraic and express the higher dimension components of $q^{1,1}$ through z-derivatives of the physical dimension ones (they leave as independent also four auxiliary fields which enter the θ expansion of $q^{1,1}$ as coefficients before the monomials $\theta^{1,0\underline{i}}\theta^{0,1\underline{a}}$).

To understand the origin of the difference between these two types of constraints, let us perform the reduction of the $(4,4)$ $SU(2)_L\times SU(2)_R$ harmonic superspace to the standard $(4,4)$ $SU(2)$ one. It is accomplished by identifying harmonic variables $u^{\pm 1\,i} = v^{\pm 1\,a}$ and, respectively, both harmonic $U(1)$ charges. The harmonic derivative $D^{(+2)}$ (2.3) is recognized as the sum of the left and right ones

$$D^{(+2)} = D^{2,0} + D^{0,2}\ .$$

From this consideration it is already clear that there is no smooth transition between the constraints (3.7) and (2.15). The field content of $q^{1,1}$ also changes. While before identifying harmonics u and v the matrix of physical bosons $q^{ia}(z)$ ($q^{1,1} = q^{ia}u^1_i v^1_a + ...$) comprises 4 independent fields, after the identification this number is reduced to 3 (only the symmetric part of q^{ia} survives). As a result of imposing the constraint (2.15) on the reduced superfield, the lost fourth scalar field reappears as a solution to the divergencelessness condition for the $2D$ vector field components multiplying the θ monomials $(\theta^{(+)+})^2$, $(\theta^{(+)-})^2$. Note that the smooth transition between the two superfield systems becomes possible in the dual action of $q^{1,1}$ (see below).

Despite the essential difference between the constraints (3.7) and (2.15), invariant actions of $q^{1,1}$ look similar to those of $l^{(+2)}$ (2.14). The general off-shell action of n superfields $q^{1,1\,M}$ ($M=1,2,...n$) reads

$$S_q = \int \mu^{-2,-2}\ L^{2,2}(q^{1,1\,M}(\zeta,u,v),\ u,\ v)\ . \tag{3.8}$$

Here

$$\mu^{-2,-2} = d^6\zeta\ [du\ dv] = d^2z\ d^2\theta^{1,0}\ d^2\theta^{0,1}\ [du\ dv]$$

is the analytic superspace integration measure. The dimensionless analytic superfield Lagrangian $L^{2,2}(q^{1,1\,M}, u_i^{\pm1}, v_a^{\pm1})$ bears in general an arbitrary dependence on its arguments, the only restriction being a compatibility with the external $U(1)$ charges $2,2$. The free action of $q^{1,1\,M}$ is given by

$$S_q^{free} \sim \int \mu^{-2,-2}\, q^{1,1\,M}\, q^{1,1\,M}\ , \tag{3.9}$$

so for consistency we are led to assume

$$\det\left(\frac{\partial^2 L^{2,2}}{\partial q^{1,1\,M}\partial q^{1,1\,N}}\right)\ |_{q^{1,1}=0} \neq 0\ . \tag{3.10}$$

For completeness, we also add the constraints on $q^{1,1\,M}(\zeta, u, v)$

$$D^{2,0}q^{1,1\,M} = D^{0,2}q^{1,1\,M} = 0\ . \tag{3.11}$$

The passing to the component form of the action is straightforward [13]. The bosonic sigma model action consists of two parts related to each other by $(4,4)$ supersymmetry: the metric part and the part including the torsion potential.

As an important particular example of $q^{1,1}$ action we give the action of $(4,4)$ extension of the group manifold $SU(2)\times U(1)$ WZNW sigma model

$$S_{wzw} = -\frac{1}{4\kappa^2}\int \mu^{-2,-2}\ \hat{q}^{1,1}\hat{q}^{(1,1)}\left(\frac{1}{(1+X)X} - \frac{\ln(1+X)}{X^2}\right)\ . \tag{3.12}$$

Here

$$\hat{q}^{1,1} = q^{1,1} - c^{1,1}\ ,\ X = c^{-1,-1}\hat{q}^{1,1}\ ,\ c^{\pm1,\pm1} = c^{ia}u_i^{\pm1}v_a^{\pm1}\ ,\ c^{ia}c_{ia} = 2\ . \tag{3.13}$$

Despite the presence of an extra quartet constant c^{ia} in the analytic superfield Lagrangian, the action (3.12) actually does not depend on c^{ia} [13] as it is invariant under arbitrary rescalings and $SU(2)\times SU(2)$ rotations of this constant.

4 Dual form of the $q^{1,1}$ action and its generalization

By adding the constraints (3.11) with the superfield Lagrange multipliers to the general $q^{1,1}$ action (3.8) one puts the latter in the form analogous to the tensor supermultiplet master action (2.13)

$$S_{q,\omega} = \int \mu^{-2,-2}\{\omega^{-1,1\,M}D^{2,0}q^{1,1\,M} + \omega^{1,-1\,M}D^{0,2}q^{1,1\,M} + L^{2,2}(q^{1,1}, u, v)\}\ . \tag{4.1}$$

The analytic superfields $q^{1,1\,M}$, $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ are now unconstrained and one can vary them to get the superfield equations of motion. Varying $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ yields the constraints (3.11) and we recover the original action (3.8). Alternatively, one can vary (4.1) with respect to $q^{1,1\,M}$, which gives rise to the equation

$$\frac{\partial L^{2,2}}{\partial q^{1,1\,M}} = D^{2,0}\omega^{-1,1\,M} + D^{0,2}\omega^{1,-1\,M} \equiv A^{1,1\,M}\ . \tag{4.2}$$

This algebraic equation is a kind of Legendre transformation expressing $q^{1,1\,M}$ as a function of $A^{1,1\,M}$

$$(4.2) \Rightarrow q^{1,1\,M} = q^{1,1\,M}(A^{1,1}, u, v)\ . \tag{4.3}$$

Substituting this expression back into (4.1), one arrives at the dual form of the $q^{(1,1)}$ action

$$\begin{aligned} S_\omega &= \int \mu^{-2,-2} L^{2,2}_\omega (A^{1,1}, u, v)\ , \\ L^{2,2}_\omega(A^{1,1}, u, v) &\equiv L^{2,2}(q^{1,1\,M}(A, u, v), u, v) - q^{1,1\,M}(A, u, v) A^{1,1\,M}\ . \end{aligned} \tag{4.4}$$

The dual action (4.4) provides a new off-shell formulation of $(4,4)$ sigma models with commuting left and right complex structures via *unconstrained* analytic $(4,4)$ superfields. The most characteristic feature of such formulations is the presence of infinite number of auxiliary fields [9, 10]. Thus, in the case at hand the physical component action for $4n$ bosons and $8n$ fermions is restored only after eliminating an infinite tower of auxiliary fields which come from the double harmonic expansion of superfields $\omega^{1,-1\,N}(\zeta, u, v)$, $\omega^{-1,1\,N}(\zeta, u, v)$.

To see in more detail how this occurs, let us focus on the bosonic degrees of freedom. The action (4.1) originally involves three independent superfields $q^{1,1\,N}$, $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$, each including $4n$ real bosonic fields in the first term of its double harmonic expansion (higher rank bosonic fields finally prove to be auxiliary and we should not care about them). Varying $q^{1,1\,N}$ yields an algebraic equation (4.2) by which $q^{1,1\,N}$ is eliminated in terms of the remaining two superfields

$$q^{1,1\,N} \sim D^{2,0}\omega^{-1,1\,N} + D^{0,2}\omega^{1,-1\,N} + ... \tag{4.5}$$

(cf. eq. (2.16)). Thereby, the number of physical dimension bosonic fields is reduced from $12n$ to $8n$. However, the number of such fields carried by two ω superfields is still twice the number of those carried by $q^{1,1}$ in the original formulation. So one may wonder how the on-shell equivalence of these two off-shell formulations is achieved. The answer is that the equivalence is guaranteed due to the invariance of the action (4.1) and its ω version (4.4) under the abelian gauge transformations

$$\delta\, \omega^{1,-1\,M} = D^{2,0}\sigma^{-1,-1\,M}\ ,\ \delta\, \omega^{-1,1\,M} = -D^{0,2}\sigma^{-1,-1\,M}\ , \tag{4.6}$$

with $\sigma^{-1,-1\,M} = \sigma^{-1,-1\,M}(\zeta, u, v)$ being arbitrary analytic functions. This gauge freedom takes away just half of the lowest superisospin multiplets in the superfields $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$, thus restoring the correct physical field content of the theory. For instance, the first components in the θ expansion of these superfields are transformed as

$$\delta\, \omega^{1,-1\,M}_0(z) = \partial^{2,0}\sigma^{-1,-1\,M}(z)\ ,\ \delta\, \omega^{-1,1\,M}_0(z) = -\partial^{0,2}\sigma^{-1,-1\,M}(z)\ , \tag{4.7}$$

and one may fix the gauge so as to entirely eliminate one set of these fields (other gauge choices are also possible). Thus, in contrast to the $q^{1,1}$ superfield formulation, where the necessary set of the physical fields is ensured by imposing the harmonic constraints on $q^{1,1}$, the same goal in the dual formulation is achieved thanks to the gauge freedom (4.6) (and after eliminating an infinite set of auxiliary fields). This gauge invariance is the

main novel feature of the dual formulation of the $q^{1,1}$ action compared to an analogous formulation of the $l^{(+2)}$ action in the conventional harmonic superspace. It is a necessary ingredient of the free action of the triple $q^{1,1\,N}$, $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$ (corresponding to the choice $L^{2,2} = q^{1,1\,N}q^{1,1\,N}$ in (4.1)) and one can expect that any reasonable generalization to the case with interaction should enjoy this important symmetry. Below we will see that this is indeed so, the abelian gauge invariance getting nonabelian in general.

For what follows it will be important to note that the gauge freedom in question reflects the commutativity of the left and right harmonic derivatives $D^{2,0}$ and $D^{0,2}$. Indeed, the equations of motion which follow by varying Lagrange multipliers $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$, viz. the constraints (3.11), are not entirely independent: due to the above commutativity they obey the evident integrability condition

$$D^{2,0}(D^{0,2}q^{1,1\,M}) - D^{0,2}(D^{2,0}q^{1,1\,M}) = 0\ . \tag{4.8}$$

In the simplest case we are considering, this condition is identically satisfied (since $L^{2,2}$ does not depend on $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$). However, in more general cases it puts non-trivial restrictions on the structure of the action. Below we will see that in all examples in which the condition (4.8) is satisfied the relevant actions respect gauge symmetry (4.6) or a nonabelian extension of it.

It is to the point here to adhere to a clarifying analogy with the abelian gauge theory in two dimensions. The harmonic derivatives $D^{2,0}, D^{0,2}$ are analogous to the x derivatives ∂_μ, $\mu = 1, 2$, two Lagrange multipliers $\omega^{1,-1\,N}$ and $-\omega^{-1,1\,N}$ being analogs of the two-dimensional $U(1)$ gauge connection A_μ (actually, of n independent copies of it), the quantity $A^{1,1\,N}$ in (4.2) an analog of the gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \epsilon_{\mu\nu}F$. Then the dual action (4.1) is analogous to the first order form of the Maxwell action of A_μ[3], while the constraints (3.7) are the precise analog of the sourceless Maxwell equation

$$\partial^\mu F_{\mu\nu} = \partial^\mu \epsilon_{\mu\nu} F = 0\ . \tag{4.9}$$

The self-consistency condition (4.8) is a counterpart of the "kinematical" conservation law

$$\partial^\nu(\partial^\mu F_{\mu\nu}) = 0\ . \tag{4.10}$$

The conservation law (4.10) ceases to be trivial after inserting a matter current into the r.h.s. of (4.9): in this case it requires the current to be conserved as a consequence of the equations of motion, which imposes severe restrictions on the structure of this current and implies the gauge symmetry of the free action to extend to the whole action (this symmetry can get nonabelian in general). Quite similarly, after allowing for a $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$ dependence in $L^{2,2}$ there will appear a non-zero "current" in the r.h.s. of eqs. (3.11) and the condition (4.8) will become the harmonic conservation law for this current, severely restricting the structure of the latter and, hence, of $L^{2,2}$. In the sequel we will sometimes resort to this analogy.

The last comment concerning transformations (4.6) is that they define a *genuine* symmetry of the actions (4.1), (4.4), contrary, e.g., to the transformations (2.4), (2.5), (2.6)

[3] Just as the dual action of $(4,4)$ supersymmetric hyper-Kähler sigma model (2.11) is analogous to the first order form of a scalar field action.

which are a kind of equivalence redefinitions of the involved superfields and potentials. These latter transformations leave the relevant actions form-invariant but change the precise structure of the potentials in them. The actions (4.1), (4.4) also possess a restricted type of such target space form-invariance. Later on we will present the explicit form of the latter for a generalization of (4.1).

Let us turn to generalizing the action (4.1). As was argued in [2, 16], with making use of the $(4,4)$ twisted supermultiplet alone one may construct only the $(4,4)$ sigma models with mutually commuting left and right complex structures. Then a natural way to approach the problem of constructing off-shell $(4,4)$ superfield actions with non-commuting structures is to seek for such generalizations of the action (4.1) which *do not admit the passing to a pure $q^{1,1}$ form*. The rest of the paper is devoted to deducing and studying such generalizations.

The action (4.1) is an analog of the dual $l^{(+2)}$ action (2.13), the triple of superfields $q^{(1,1)}$, $\omega^{1,-1}$, $\omega^{1,-1}$ being an analog of the pair $l^{(+2)}, \omega$. So one may write the most general action of this triple, making in (4.1) the substitutions like those which lead from (2.13) to the general l, ω action (2.11). In this way one obtains

$$
\begin{aligned}
S_{q,\omega} &= \int \mu^{-2,-2}\{H^{2,2} + H^{-1,1\,M}D^{2,0}q^{1,1\,M} + H^{1,-1\,M}D^{0,2}q^{1,1\,M} + H^{1,1\,M}D^{0,2}\omega^{1,-1\,M} \\
&\quad + \tilde{H}^{1,1\,M}D^{2,0}\omega^{-1,1\,M} + H^{-1,3\,M}D^{2,0}\omega^{1,-1\,M} + H^{3,-1\,M}D^{0,2}\omega^{-1,1\,M}\} \\
&\equiv \int \mu^{-2,-2}\mathcal{L}^{2,2}_{q,\omega}(q,\omega,u,v)\,, \qquad (4.11)
\end{aligned}
$$

where *a priori* all the potentials H are arbitrary functions of the superfields $q^{1,1\,M}$, $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ and harmonics u, v. For the time being we leave aside the important question of implementing the gauge freedom (4.6) in this action and will try to use the set of invariances of the type (2.4), (2.6) to reduce the number of independent potentials as much as possible.

One type of such invariances of the action (4.11) is related to reparametrizations of the involved superfields

$$
\begin{aligned}
\delta q^{1,1\,M} &= \Lambda^{1,1\,M}(q,\omega,u,v)\,, \quad \delta\omega^{1,-1\,M} = \Lambda^{1,-1\,M}(q,\omega,u,v)\,, \\
\delta\omega^{-1,1\,M} &= \Lambda^{-1,1\,M}(q,\omega,u,v)\,. \qquad (4.12)
\end{aligned}
$$

It is straightforward to find the transformations of the potentials such that the action is form-invariant. Their explicit structure is not too enlightening.

Another type of invariance is similar to the hyper-Kähler one (2.6) and is related to the freedom of adding full harmonic deriavtives to the superfield Lagrangian in (4.11)

$$
\begin{aligned}
&\mathcal{L}^{2,2}_{q,\omega} \Rightarrow \mathcal{L}^{2,2}_{q,\omega} + D^{2,0}\Lambda^{0,2} + D^{0,2}\Lambda^{2,0}\,, \qquad (4.13) \\
&\Lambda^{2,0} = \Lambda^{2,0}(q,\omega,u,v)\,, \quad \Lambda^{0,2} = \Lambda^{0,2}(q,\omega,u,v)\,.
\end{aligned}
$$

Once again, it is easy to indicate how the potentials should transform to generate the shifts (4.13). It will be important for our consideration that, assuming the existence of the flat limit (given by the action (4.1) with $L^{2,2}(q,u,v) = q^{1,1\,N}q^{1,1\,N}$), the full gauge freedom (4.12), (4.13) can be fixed so that

$$
\begin{aligned}
H^{-1,1\,N} &= \alpha\omega^{-1,1\,N}\,, \quad H^{1,-1\,N} = \beta\omega^{1,-1\,N}\,, \\
H^{1,1\,N} &= (1+\beta)q^{1,1\,N}\,, \quad \tilde{H}^{1,1\,N} = (1+\alpha)q^{1,1\,N} + \hat{H}^{1,1\,N}\,, \qquad (4.14)
\end{aligned}
$$

α, β being arbitrary parameters. In this gauge (which is an analog of the gauges (2.7), (2.12)) the action still contains four independent potentials, $H^{2,2}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$ and $\hat{H}^{1,1\,N}$,

$$
\begin{aligned}
S_{q,\omega} \;=\; & \int \mu^{-2,-2}\{q^{1,1\,M}D^{0,2}\omega^{1,-1\,M} + (q^{1,1\,M} + \hat{H}^{1,1\,M})D^{2,0}\omega^{-1,1\,M} \\
& + H^{-1,3\,M}D^{2,0}\omega^{1,-1\,M} + H^{3,-1\,M}D^{0,2}\omega^{-1,1\,M} + H^{2,2}\}\,, \qquad (4.15)
\end{aligned}
$$

and is invariant under the following target space gauge transformations which are a mixture of (4.12) and (4.13) (the unconstrained analytic parameters $\Lambda^{2,0}, \Lambda^{0,2}$ below do not precisely coincide with those in eq. (4.13), but are related to them in a simple way)

$$
\begin{aligned}
\delta\hat{H}^{1,1\,M} &= -\Lambda^{1,1\,M} + \frac{\partial\Lambda^{0,2}}{\partial\omega^{-1,1\,M}} + \Lambda^{1,-1\,N}\frac{\partial H^{-1,3\,N}}{\partial\omega^{-1,1\,M}} + \Lambda^{-1,1\,N}\frac{\partial\hat{H}^{1,1\,N}}{\partial\omega^{-1,1\,M}} \\
\delta H^{-1,3\,M} &= \frac{\partial\Lambda^{0,2}}{\partial\omega^{1,-1\,M}} + \Lambda^{1,-1\,N}\frac{\partial H^{-1,3\,N}}{\partial\omega^{1,-1\,M}} + \Lambda^{-1,1\,N}\frac{\partial\hat{H}^{1,1\,N}}{\partial\omega^{1,-1\,M}} \\
\delta H^{3,-1\,M} &= \frac{\partial\Lambda^{2,0}}{\partial\omega^{-1,1\,M}} + \Lambda^{-1,1\,N}\frac{\partial H^{3,-1\,N}}{\partial\omega^{-1,1\,M}} \\
\delta H^{2,2} &= \partial^{2,0}\Lambda^{0,2} + \partial^{0,2}\Lambda^{2,0} + \Lambda^{1,-1\,N}\partial^{2,0}H^{-1,3\,N} \\
&\quad + \Lambda^{-1,1\,N}(\partial^{2,0}\hat{H}^{1,1\,N} + \partial^{0,2}H^{3,-1\,N}) \qquad (4.16)
\end{aligned}
$$

with

$$
\begin{aligned}
\Lambda^{1,1\,M} &= \frac{\partial\Lambda^{2,0}}{\partial\omega^{1,-1\,M}} - (B^{-1})^{FN}\frac{\partial H^{3,-1\,N}}{\partial\omega^{1,-1\,M}}\left\{\frac{\partial\Lambda^{0,2}}{\partial q^{1,1\,F}} - \frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,T}}\frac{\partial H^{-1,3\,T}}{\partial q^{1,1\,F}}\right\} \\
\Lambda^{1,-1\,M} &= -\frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,M}} + (B^{-1})^{FN}\frac{\partial H^{3,-1\,N}}{\partial q^{1,1\,M}}\left\{\frac{\partial\Lambda^{0,2}}{\partial q^{1,1\,F}} - \frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,T}}\frac{\partial H^{-1,3\,T}}{\partial q^{1,1\,F}}\right\} \\
\Lambda^{-1,1\,M} &= -(B^{-1})^{NM}\left\{\frac{\partial\Lambda^{0,2}}{\partial q^{1,1\,N}} - \frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,T}}\frac{\partial H^{-1,3\,T}}{\partial q^{1,1\,N}}\right\} \qquad (4.17) \\
B^{MN} &= \delta^{MN} + \frac{\partial\hat{H}^{1,1\,M}}{\partial q^{1,1\,N}} - \frac{\partial H^{3,-1\,M}}{\partial q^{1,1\,F}}\frac{\partial H^{-1,3\,F}}{\partial q^{1,1\,N}}\,,\; B^{MN}(B^{-1})^{NL} = \delta^{ML}
\end{aligned}
$$

(one should add, of course, the coordinate transformations (4.12) with the parameters (4.17)). Note that in the case of general manifold ($M = 1, 2...n, n > 1$) it is impossible to gauge away any of the surviving potentials with the help of this remaining gauge freedom, though one can still put them in the form similar to the normal gauge of the hyper-Kähler potential $L^{(+4)}$ [7]. The fact that there remain three more potentials besides $H^{2,2}$ (which is a direct analog of $L^{(+4)}$) is the essential difference of the considered case with torsion from the torsionless hyper-Kähler case. It is worth mentioning that upon the reduction to the $(4,4)$ $SU(2)$ harmonic superspace the superfields $\omega^{1,-1\,N}$ and $\omega^{-1,1\,N}$ in (4.11) are identified with each other and recognized as the single superfield ω^N, $q^{1,1\,N} \Rightarrow l^{(+2)\,N}$, $H^{2,2} \Rightarrow L^{(+4)}$, and the potentials $\hat{H}^{1,1\,N}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$ are combined into a shift of $l^{(+2)\,N}$. This shift can be absorbed in an equivalence redefinition of $l^{(+2)\,N}$, after which one recovers the ω, l action (2.11) of the general $(4,4)$ hyper-Kähler sigma model in the "flat" gauge (2.12). Note that the potentials in (4.11), (4.15) will turn out to be severely

constrained, so the reduction just mentioned actually produces some particular class of hyper-Kähler $(4,4)$ actions.

As was noticed in Sect.2, the $q^{(+)}$ equation of motion (2.8) following from the general $q^{(+)}$ action (2.2) has a transparent interpretation within the relevant analytic target space geometry: it expresses the vielbein $E^{(+3)\,M} \equiv D^{(+2)}q^{(+)\,M}$ of the analytic target space harmonic derivative via the unconstrained hyper-Kähler potentials $L^{(+4)}$, $L^{(+)}_M$. At present we have no clear understanding which kind of the analytic target space geometry underlies the general off-shell $(4,4)$ action with torsion (4.11). By analogy with the hyper-Kähler case, studying this action, the involved objects and their equations of motion could help to clarify this point.

We will deal with the gauge-fixed action (4.15). The equations of motion following from it read

$$
\begin{aligned}
& D^{0,2}\omega^{1,-1\,M} + D^{2,0}\omega^{-1,1\,N}\left(\delta^{NM} + \frac{\partial \hat{H}^{1,1\,N}}{\partial q^{1,1\,M}}\right) + D^{0,2}\omega^{-1,1\,N}\frac{\partial H^{3,-1\,N}}{\partial q^{1,1\,M}} \\
& + D^{2,0}\omega^{1,-1\,N}\frac{\partial H^{-1,3\,N}}{\partial q^{1,1\,M}} = -\frac{\partial H^{2,2}}{\partial q^{1,1\,M}}\,, \\
& D^{0,2}q^{1,1\,M} + D^{2,0}q^{1,1\,N}\frac{\partial H^{-1,3\,M}}{\partial q^{1,1\,N}} + D^{2,0}\omega^{1,-1\,N}\left(\frac{\partial H^{-1,3\,M}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{1,-1\,M}}\right) \\
& + D^{2,0}\omega^{-1,1\,N}\left(\frac{\partial H^{-1,3\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial \hat{H}^{1,1\,N}}{\partial \omega^{1,-1\,M}}\right) \\
& - D^{0,2}\omega^{-1,1\,N}\frac{\partial H^{3,-1\,N}}{\partial \omega^{1,-1\,M}} = \frac{\partial H^{2,2}}{\partial \omega^{1,-1\,M}} - \partial^{2,0}H^{-1,3\,M}\,, \\
& D^{2,0}q^{1,1\,N}\left(\delta^{MN} + \frac{\partial \hat{H}^{1,1\,M}}{\partial q^{1,1\,N}}\right) + D^{0,2}q^{1,1\,N}\frac{\partial H^{3,-1\,M}}{\partial q^{1,1\,N}} \\
& + D^{2,0}\omega^{-1,1\,N}\left(\frac{\partial \hat{H}^{1,1\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial \hat{H}^{1,1\,N}}{\partial \omega^{-1,1\,M}}\right) + D^{2,0}\omega^{1,-1\,N}\left(\frac{\partial \hat{H}^{1,1\,M}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{-1,1\,M}}\right) \\
& + D^{0,2}\omega^{-1,1\,N}\left(\frac{\partial H^{3,-1\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial H^{3,-1\,N}}{\partial \omega^{-1,1\,M}}\right) \\
& + D^{0,2}\omega^{1,-1\,N}\frac{\partial H^{3,-1\,M}}{\partial \omega^{1,-1\,N}} = \frac{\partial H^{2,2}}{\partial \omega^{-1,1\,M}} - \partial^{2,0}\hat{H}^{1,1\,M} - \partial^{0,2}H^{3,-1\,M}. \qquad (4.18)
\end{aligned}
$$

After expressing $D^{0,2}\omega^{1,-1\,M}$ from the first of these equations

$$
\begin{aligned}
D^{0,2}\omega^{1,-1\,M} \;=\;& -\frac{\partial H^{2,2}}{\partial q^{1,1\,M}} - \left(\delta^{NM} + \frac{\partial \hat{H}^{1,1\,N}}{\partial q^{1,1\,M}}\right)D^{2,0}\omega^{-1,1\,N} \\
& -\frac{\partial H^{3,-1\,N}}{\partial q^{1,1\,M}}D^{0,2}\omega^{-1,1\,N} - \frac{\partial H^{-1,3\,N}}{\partial q^{1,1\,M}}D^{2,0}\omega^{1,-1\,N}\,, \qquad (4.19)
\end{aligned}
$$

the remaining two can be cast in the form

$$
\begin{aligned}
D^{0,2}q^{1,1\,M} \;=\;& T^{1,3\,M} + T^{0,2\,NM}\,D^{2,0}\omega^{-1,1\,N} + T^{2,0\,NM}\,D^{0,2}\omega^{-1,1\,N} \\
& + T^{-2,4\,NM}\,D^{2,0}\omega^{1,-1\,N} \equiv J^{1,3\,M} \qquad (4.20) \\
D^{2,0}q^{1,1\,M} \;=\;& G^{3,1\,M} + G^{2,0\,NM}\,D^{2,0}\omega^{-1,1\,N} + G^{4,-2\,NM}\,D^{0,2}\omega^{-1,1\,N} \\
& + G^{0,2\,NM}\,D^{2,0}\omega^{1,-1\,N} \equiv J^{3,1\,M}\,. \qquad (4.21)
\end{aligned}
$$

Here, the coefficient functions depend only on the potentials and their derivatives. It is easy to explicitly find these functions. We leave this for the reader as a useful exercise.

To realize the geometric meaning of the above equations, let us compare the case under consideration with the hyper-Kähler one. A direct analog of the analytic harmonic target space $(l^{(+2)\,\alpha}, \omega^\alpha, u_i^{(\pm)})$ in the present case is the manifold spanned by the coordinates $q^{1,1\,N}$, $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$ and the target harmonics $u_i^{\pm 1}$, $v_a^{\pm 1}$. Let us introduce, like in the hyper-Kähler case (eq. (2.9)), the target space harmonic derivatives $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$. When acting on the analytic subspace coordinates u, v, $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$, they are given by the expressions

$$
\begin{aligned}
\mathcal{D}^{2,0} &= \partial^{2,0} + E^{3,1\,M}\frac{\partial}{\partial q^{1,1\,M}} + \tilde{E}^{1,1\,M}\frac{\partial}{\partial \omega^{-1,1\,M}} + E^{3,-1\,M}\frac{\partial}{\partial \omega^{1,-1\,M}}\,, \\
\mathcal{D}^{0,2} &= \partial^{0,2} + E^{1,3\,M}\frac{\partial}{\partial q^{1,1\,M}} + E^{-1,3\,M}\frac{\partial}{\partial \omega^{-1,1\,M}} + E^{1,1\,M}\frac{\partial}{\partial \omega^{1,-1\,M}}\,, \qquad (4.22)
\end{aligned}
$$

$$
\begin{aligned}
E^{3,1\,M} &\equiv D^{2,0}q^{1,1\,M}\,, \quad E^{1,1\,M} \equiv D^{0,2}\omega^{1,-1\,M}\,, \quad E^{3,-1\,M} \equiv D^{2,0}\omega^{1,-1\,M} \\
E^{1,3\,M} &\equiv D^{0,2}q^{1,1\,M}\,, \quad E^{-1,3\,M} \equiv D^{0,2}\omega^{-1,1\,M}\,, \quad \tilde{E}^{1,1\,M} \equiv D^{2,0}\omega^{-1,1\,M}\,, \qquad (4.23)
\end{aligned}
$$

where $\partial^{2,0}$, $\partial^{0,2}$ act only on the "target" harmonics, i.e. those appearing explicitly in the potentials and other geometric objects. Thus the harmonic derivatives of the involved analytic superfields acquire the geometric meaning of vielbeins covariantizing the flat derivatives $\partial^{2,0}$, $\partial^{0,2}$ with respect to the analytic target space gauge group (4.16), (4.17). When promoted to the full target space, $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$ can get extra pieces containing additional partial derivatives contracted with the proper vielbeins (e.g., one may expect that the full harmonic target space in the analytic basis involves, in addition to the triple of the analytic subspace coordinates $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$, one more coordinate $l^{-1,-1\,N}$ which is represented by a general harmonic superfield). In what follows we will never specify the complete structure of $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$ and simply assume that they have the proper action on all the objects depending on harmonics u and v. In particular, when acting on an arbitrary analytic harmonic $(4,4)$ superfield (it can be, e.g., a local function of superfields $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$ and explicitly include harmonics u and v), they always coincide with $D^{2,0}$ and $D^{0,2}$.

Keeping in mind the definition (4.23), the equations of motion (4.18) (or their equivalent form (4.19) - (4.21)) can be interpreted in a geometric way as the relations expressing some of the harmonic vielbeins via the potentials $H^{2,2}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$, $\hat{H}^{1,1\,N}$. One immediately realizes what is the main difference from the hyper-Kähler relation (2.8). Only three harmonic vielbeins, namely, $E^{1,3\,N}$, $E^{3,1\,N}$ and some linear combination of $\tilde{E}^{1,1\,N}$, $E^{1,1\,N}$ (in (4.19) we have chosen it to coincide with $E^{1,1\,N}$), are really eliminated. Three remaining vielbeins, $\tilde{E}^{1,1\,N}$, $E^{1,-3\,N}$ and $E^{3,-1\,N}$, are not constrained by these equations and so should be treated at this step as some independent quantities. One cannot even conclude that they are local functions of the analytic target space coordinates u, v, $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$.

In the flat target space limit (with $H^{2,2} \sim q^{1,1\,M}q^{1,1\,M}$ and all other potentials vanishing) these superfluous vielbeins besides $\hat{H}^{1,1}$ drop out from the equations of motion. Then it seems natural and tempting to assume that in the case with interaction they do not contribute as well, i.e. the corresponding coefficients in eqs. (4.19) - (4.21) vanish.

This is indeed so and comes about as the result of taking account of the compatibility relations which follow from the obvious commutativity condition

$$[\mathcal{D}^{2,0}, \mathcal{D}^{0,2}] = 0 \Rightarrow \tag{4.24}$$

$$\begin{aligned}
\mathcal{D}^{0,2} E^{3,1\,N} - \mathcal{D}^{2,0} E^{1,3\,N} &= 0 & (4.25)\\
\mathcal{D}^{2,0} E^{-1,3\,N} - \mathcal{D}^{0,2} E^{1,1\,N} &= 0 & (4.26)\\
\mathcal{D}^{0,2} E^{3,-1\,N} - \mathcal{D}^{2,0} \tilde{E}^{1,1\,N} &= 0\,. & (4.27)
\end{aligned}$$

These relations are identically satisfied with the definition (4.23) (from the point of view of the target space geometry these special expressions for the harmonic vielbeins mean that the latter are induced as a result of passing to the analytic basis in the target space from some central basis where the harmonic derivatives are short, $\mathcal{D}^{2,0} = \partial^{2,0}$, $\mathcal{D}^{0,2} = \partial^{0,2}$). However, once the vielbeins are subjected to the dynamical equations (4.19) - (4.21), these relations become non-trivial consistency conditions on the potentials $H^{2,2}$, $H^{3,1}$, $H^{1,3}$, $\hat{H}^{1,1}$. Indeed, eq. (4.25) together with eqs. (4.21), (4.20) implies the integrability condition

$$D^{2,0} J^{1,3\,M} - D^{0,2} J^{3,1\,M} = 0\,, \tag{4.28}$$

which severely constrains the coefficient functions T and G in $J^{1,3\,M}$, $J^{3,1\,M}$ and, further, the potentials through which these functions are expressed. In Sect.6 we will show that these constraints, being combined with the target space gauge freedom (4.16), (4.17), allows one to get rid of all the potentials in the action (4.15) except for $H^{2,2}$. Note that two other relations in the set (4.25) - (4.27) do not place any restrictions on the potentials, as is seen from the structure of eqs. (4.19) - (4.21).

In order to gain some experience, we will first consider the $n = 1$ case.

5 Digression: $n = 1$ example

In this case we deal with one triple of analytic superfields $q^{1,1}$, $\omega^{1,-1}$, $\omega^{-1,1}$ and four-dimensional manifold of physical bosons (providing the gauge freedom (4.6) or some its generalization hold). The action (4.15) can be further simplified because the potentials $H^{-1,3}$, $H^{3,-1}$ become pure gauge

$$H^{-1,3} = H^{3,-1} = 0 \tag{5.1}$$

$$\Rightarrow\; S^{(1)}_{q,\omega} = \int \mu^{-2,-2} \{ q^{1,1} D^{0,2} \omega^{1,-1} + (q^{1,1} + \hat{H}^{1,1}) D^{2,0} \omega^{-1,1} + H^{2,2} \}\,. \tag{5.2}$$

Thus, the general action of the triple $q^{1,1}$, $\omega^{1,-1}$, $\omega^{-1,1}$ is characterized by two potentials $H^{2,2} = H^{2,2}(q,\omega,u,v)$ and $\hat{H}^{1,1} = \hat{H}^{1,1}(q,\omega,u,v)$ which, before enforcing the integrability condition (4.28), are arbitrary functions of their arguments. The action is still invariant under the restricted class of reparametrizations preserving the gauge (5.1)

$$\delta \hat{H}^{1,1} = -\Lambda^{1,1} + \frac{\partial \Lambda^{0,2}}{\partial \omega^{-1,1}} + \Lambda^{-1,1} \frac{\partial \hat{H}^{1,1}}{\partial \omega^{-1,1}}\,,\; \delta H^{2,2} = \partial^{2,0} \Lambda^{0,2} + \partial^{0,2} \Lambda^{2,0} + \Lambda^{-1,1} \partial^{2,0} \hat{H}^{1,1} \tag{5.3}$$

$$\delta q^{1,1} \;\equiv\; \Lambda^{1,1} = \frac{\partial \Lambda^{2,0}}{\partial \omega^{1,-1}}\,,\; \delta \omega^{1,-1} \;\equiv\; \Lambda^{1,-1} \;=\; -\frac{\partial \Lambda^{2,0}}{\partial q^{1,1}}\,,$$

$$\delta \omega^{-1,1} \;\equiv\; \Lambda^{-1,1} \;=\; -\left(1+\frac{\partial \hat{H}^{1,1}}{\partial q^{1,1}}\right)^{-1} \frac{\partial \Lambda^{0,2}}{\partial q^{1,1}} \;\equiv\; -B^{-1}\frac{\partial \Lambda^{0,2}}{\partial q^{1,1}} \tag{5.4}$$

$$\frac{\partial \Lambda^{2,0}}{\partial \omega^{-1,1}} = 0 \Rightarrow \Lambda^{2,0} = \Lambda^{2,0}(q^{1,1},\omega^{1,-1},u,v)\,,\;\; \frac{\partial \Lambda^{0,2}}{\partial \omega^{1,-1}} - B^{-1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}}\frac{\partial \Lambda^{0,2}}{\partial q^{1,1}} = 0\,. \tag{5.5}$$

The set of equations (4.18) is also essentially simplified

$$D^{0,2}\omega^{1,-1} \;=\; -\frac{\partial H^{2,2}}{\partial q^{1,1}} - D^{2,0}\omega^{-1,1}B\,,\;\; D^{0,2}q^{1,1} \;=\; \frac{\partial H^{2,2}}{\partial \omega^{1,-1}} + D^{2,0}\omega^{-1,1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}}\,,$$

$$D^{2,0}q^{1,1} \;=\; B^{-1}\left(\frac{\partial H^{2,2}}{\partial \omega^{-1,1}} - \partial^{2,0}\hat{H}^{1,1} - D^{2,0}\omega^{1,-1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}}\right)\,. \tag{5.6}$$

To extract the consequences of the integrability condition (4.28), we act on the r.h.s. of the second and third equations in (5.6) by $D^{2,0}$ and $D^{0,2}$, use once again (5.6) to eliminate $D^{0,2}q^{1,1}$, $D^{2,0}q^{1,1}$ and $D^{0,2}\omega^{1,-1}$, and finally equate the obtained expressions. Equating, in both sides of the resulting equality, the terms without harmonic derivatives, as well as the coefficients before the independent structures which are $D^{2,0}\omega^{1,-1}$, $D^{0,2}\omega^{-1,1}$, $D^{2,0}\omega^{-1,1}$, all possible products of them, and $(D^{2,0})^2\omega^{1,-1}$, $(D^{0,2})^2\omega^{-1,1}$, $D^{2,0}D^{0,2}\omega^{-1,1}$, $(D^{2,0})^2\omega^{-1,1}$, we arrive at the set of constraints on the potentials $H^{2,2}$ and $\hat{H}^{1,1}$. Since we started from the equations (5.6) which respect the residual target space gauge freedom (5.3) - (5.5), the set of integrability constraints is also covariant. They look rather ugly even in the $n=1$ case, so for the time being we do not write down all of them. We firstly consider the most simple one following from equating to zero the coefficient before the product $(D^{2,0}\omega^{1,-1})\,(D^{0,2}\omega^{-1,1})$ in the $n=1$ version of (4.28).

It reads

$$\frac{\partial G^{0,2}}{\partial \omega^{-1,1}} = 0\,,\;\; \left(G^{0,2} \equiv B^{-1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}}\right)\,. \tag{5.7}$$

It is straightforward to check that this condition is covariant under the *whole* target space gauge group (5.3) - (5.5). Further, $G^{0,2}$ transforms as

$$\delta G^{0,2} = -\frac{\partial^2 \Lambda^{2,0}}{\partial \omega^{1,-1}\partial \omega^{1,-1}} + ...\,, \tag{5.8}$$

where dots stand for field-dependent terms. Taking into account that both $G^{0,2}$ and $\Lambda^{2,0}$ do not depend on $\omega^{-1,1}$ and eq. (5.7) is covariant, we conclude from (5.8) that $G^{0,2}$ can be gauged away, thus giving rise to the gauge-fixing condition:

$$G^{0,2} = 0 \;\;\Rightarrow\;\; \frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}} = 0 \;\;\Rightarrow \hat{H}^{1,1} = \hat{H}^{1,1}(q^{1,1},\omega^{-1,1},u,v)\,. \tag{5.9}$$

The residual target space gauge freedom of (5.9) is given by the transformations (5.3) - (5.5) with the following additional restrictions on the parameters

$$\frac{\partial^2 \Lambda^{2,0}}{\partial \omega^{1,-1}\partial \omega^{1,-1}} \;=\; 0 \Rightarrow \Lambda^{2,0} = \lambda^{2,0}(q,u,v) + \omega^{1,-1}\lambda^{1,1}(q,u,v)\,,$$

$$\frac{\partial \Lambda^{0,2}}{\partial \omega^{1,-1}} \;=\; 0 \Rightarrow \Lambda^{0,2} = \Lambda^{0,2}(q^{1,1},\omega^{-1,1},u,v)\,. \tag{5.10}$$

Keeping in mind that $\hat{H}^{1,1}$ does not depend on $\omega^{1,-1}$, this gauge freedom is sufficient to entirely gauge away $\hat{H}^{1,1}$

$$\hat{H}^{1,1}=0\ . \tag{5.11}$$

The gauge-fixed action is still invariant under the transformations (5.3), (5.4) with

$$\Lambda^{2,0}=\lambda^{2,0}(q,u,v)+\omega^{1,-1}\lambda^{1,1}(q,u,v)\ ,\quad \Lambda^{0,2}=\lambda^{0,2}(q,u,v)+\omega^{-1,1}\lambda^{1,1}(q,u,v)\ . \tag{5.12}$$

The set of equations (5.6) becomes

$$D^{0,2}\omega^{1,-1}+D^{2,0}\omega^{-1,1}=-\frac{\partial H^{2,2}}{\partial q^{1,1}}\ ,\quad D^{0,2}q^{1,1}=\frac{\partial H^{2,2}}{\partial \omega^{1,-1}}\ , D^{2,0}q^{1,1}=\frac{\partial H^{2,2}}{\partial \omega^{-1,1}}\ . \tag{5.13}$$

Now it is easy to show that the remainder of the integrability constraints is reduced to the four conditions

$$\frac{\partial^2 H^{2,2}}{\partial\omega^{-1,1}\partial\omega^{-1,1}}=\frac{\partial^2 H^{2,2}}{\partial\omega^{1,-1}\partial\omega^{1,-1}}=\frac{\partial^2 H^{2,2}}{\partial\omega^{1,-1}\partial\omega^{-1,1}}=0\ , \tag{5.14}$$

$$\left(\partial^{2,0}+\frac{\partial H^{2,2}}{\partial\omega^{-1,1}}\frac{\partial}{\partial q^{1,1}}\right)\frac{\partial H^{2,2}}{\partial\omega^{1,-1}}-\left(\partial^{0,2}+\frac{\partial H^{2,2}}{\partial\omega^{1,-1}}\frac{\partial}{\partial q^{1,1}}\right)\frac{\partial H^{2,2}}{\partial\omega^{-1,1}}=0\ . \tag{5.15}$$

From eqs. (5.14) we find

$$H^{2,2}(u,v,q,\omega)=h^{2,2}(u,v,q)+\omega^{1,-1}h^{1,3}(u,v,q)+\omega^{-1,1}h^{3,1}(u,v,q)\ , \tag{5.16}$$

after which eq. (5.15) can be rewritten as

$$\left(\partial^{2,0}+h^{3,1}\frac{\partial}{\partial q^{1,1}}\right)h^{1,3}-\left(\partial^{0,2}+h^{1,3}\frac{\partial}{\partial q^{1,1}}\right)h^{3,1}\equiv\nabla^{2,0}h^{1,3}-\nabla^{0,2}h^{3,1}=0\ . \tag{5.17}$$

The action and constraints are covariant under the transformations (5.3), (5.4) with the restricted parameters (5.12)

$$\delta h^{2,2}=\nabla^{2,0}\lambda^{0,2}+\nabla^{0,2}\lambda^{2,0}\ ,\quad \delta h^{1,3}=\nabla^{0,2}\lambda^{1,1}\ ,\quad \delta h^{3,1}=\nabla^{2,0}\lambda^{1,1}\ . \tag{5.18}$$

It is easy to see that the action, with taking account of the constraint (5.17), is invariant under the following generalization of gauge transformations (4.6)

$$\delta\omega^{1,-1}=\left(D^{2,0}+\frac{\partial h^{3,1}}{\partial q^{1,1}}\right)\sigma^{-1,-1}\ ,\quad \delta\omega^{-1,1}=-\left(D^{0,2}+\frac{\partial h^{1,3}}{\partial q^{1,1}}\right)\sigma^{-1,-1}\ , \tag{5.19}$$

and so propagates 4 bosonic fields like the action (4.1).

Despite the appearance of nonlinearities, these transformations, like (4.6), are abelian and this property already suggests that the action (5.2) with the gauge condition (5.11) is actually a reparametrization of the dual form of the $q^{1,1}$ action (4.1). This is indeed so. It is easy to show (starting with a linearized level) that the general solution to the constraint (5.17) is given by

$$h^{1,3}=\nabla^{0,2}\Sigma^{1,1}(u,v,q)\ ,\quad h^{3,1}=\nabla^{2,0}\Sigma^{1,1}(u,v,q)\ , \tag{5.20}$$

with $\Sigma^{1,1}(u,v,q)$ being an arbitrary function (the covariant derivatives $\nabla^{2,0}$, $\nabla^{0,2}$ commute as a consequence of (5.17)). Then we can make use of the invariance (5.18) to entirely gauge away $h^{1,3}$ and $h^{3,1}$.

Thus, after fixing gauges with respect to the target space reparametrizations and employing the consequences of the integrability condition (4.25), the general $n=1$ action (5.2) coincides, modulo a field redefinition, with the general dual action (4.1) of one self-interacting twisted $(4,4)$ multiplet. So the relevant $(4,4)$ sigma models always admit a formulation in terms of single twisted superfield $q^{1,1}$ (constrained by (3.7)) and, in accord with the arguments of Refs. [2, 16], correspond to the case of mutually commuting left and right complex structures on the target. In the next Section we will show that, beginning with $n=2$, this equivalence to the action (4.1) ceases to hold in general.

6 Back to the general case

In solving the integrability constraint (4.28) for the general case with $n>1$ we will keep to the same strategy as in the $n=1$ example. Namely, we eliminate the harmonic derivatives $D^{2,0}q^{1,1\,M}$, $D^{0,2}q^{1,1\,M}$, $D^{0,2}\omega^{1,-1\,M}$ in (4.28) in terms of the remaining ones with the help of equations of motion (4.19) - (4.21) and, after this, equate to zero the coefficients before independent structures. In this way we get a set of constraints on the potentials H which is by construction covariant under the target space reparametrization group (4.16), (4.17). Some of these constraints are covariant on their own, while others are mixed under (4.16). Instead of writing down the full set of constraints, we will first discuss a few selected ones and show that they, being combined with the gauge freedom (4.16), (4.17), essentially reduce the number of independent potentials. This will allow us to present the remainder of the integrability constraints in a concise form.

As a first step we write down the constraint following from nullifying the coefficient before $(D^{0,2})^2\omega^{-1,1\,M}$

$$F^{4,-2\,[M,N]} \equiv \frac{\partial H^{3,-1\,M}}{\partial \omega^{-1,1\,N}} + \frac{\partial H^{3,-1\,M}}{\partial q^{1,1\,S}}\frac{\partial H^{3,-1\,N}}{\partial \omega^{1,-1\,S}} - (M \leftrightarrow N) = 0\ . \tag{6.1}$$

It is not difficult to verify that this constraint is covariant with respect to (4.16), (4.17)

$$\delta F^{4,-2\,[M,N]} = \left(\frac{\partial \Lambda^{-1,1\,T}}{\partial \omega^{-1,1\,M}} + \frac{\partial \Lambda^{-1,1\,T}}{\partial q^{1,1\,S}}\frac{\partial H^{3,-1\,M}}{\partial \omega^{1,-1\,S}}\right) F^{4,-2\,[T,N]} - (M \leftrightarrow N)\ . \tag{6.2}$$

Then it immediately follows that $H^{3,-1\,M}$ can be completely eliminated. Indeed, using gauge freedom (4.16), one can gauge away the totally symmetric parts of all the coefficients in the Taylor expansion of $H^{3,-1}$ in $\omega^{-1,1\,N}$. The remaining parts with mixed symmetry are zero in virtue of (6.1). Thus

$$H^{3,-1\,M} = 0\ , \tag{6.3}$$

and the gauge function $\Lambda^{2,0}$ in (4.16), (4.17) gets restricted in the following way

$$\frac{\partial \Lambda^{2,0}}{\partial \omega^{-1,1\,M}} = 0 \ \ \Rightarrow \Lambda^{2,0} = \Lambda^{2,0}(q^{1,1}, \omega^{1,-1}, u, v)\ . \tag{6.4}$$

With taking account of (6.3), the constraints which follow from vanishing of the coefficients before $D^{0,2}D^{2,0}\omega^{-1,1\,N}$, $(D^{2,0})^2\omega^{-1,1\,N}$ and $(D^{2,0})^2\omega^{1,-1\,N}$ in (4.28) are, respectively, of the form

$$
\begin{aligned}
F^{2,0\,[M,N]} &\equiv \frac{\partial \hat{H}^{1,1\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial \hat{H}^{1,1\,N}}{\partial \omega^{-1,1\,M}} = 0 & (6.5)\\
F^{0,2\,[M,N]} &\equiv \left(B^{-1}\right)^{MS}\left(\frac{\partial \hat{H}^{1,1\,S}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{-1,1\,S}}\right) - (M \leftrightarrow N) = 0 & (6.6)\\
F^{-2,4\,[M,N]} &\equiv \frac{\partial H^{-1,3\,M}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{1,-1\,M}} = 0\,. & (6.7)
\end{aligned}
$$

We will also need the constraint which comes from putting to zero the coefficient in front of the product $\left(D^{2,0}\omega^{1,-1\,N}\right)\left(D^{0,2}\omega^{-1,1\,K}\right)$

$$
\frac{\partial}{\partial \omega^{-1,1\,K}}\left\{\left(B^{-1}\right)^{ML}\left(\frac{\partial \hat{H}^{1,1\,L}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{-1,1\,L}}\right)\right\} = 0\,. \tag{6.8}
$$

(this is the $n > 1$ analog of the condition (5.7)).

The constraint (6.7) together with the gauge freedom associated with the parameter $\Lambda^{0,2}$ (still unrestricted) allow one to fully eliminate $H^{-1,3\,M}$

$$
H^{-1,3\,M} = 0\,. \tag{6.9}
$$

Since the expression in the curly brackets in (6.8) does not depend on $\omega^{-1,1\,M}$, and its transformation law starts with the symmetric inhomogeneous term

$$
-\frac{\partial^2 \Lambda^{2,0}}{\partial \omega^{1,-1\,M}\partial \omega^{1,-1\,N}}\,,
$$

the part of this expression which is symmetric in the indices M, N can be gauged away. Then the constraint (6.6) requires the antisymmetric part also to vanish, whence

$$
\frac{\partial \hat{H}^{1,1\,M}}{\partial \omega^{1,-1\,N}} = 0\,. \tag{6.10}
$$

Finally, since $\hat{H}^{1,1\,M}$ does not depend on $\omega^{1,-1\,N}$, the residual target space gauge freedom supplemented with the constraint (6.5) is still capable to completely gauge away $\hat{H}^{1,1\,M}$

$$
\hat{H}^{1,1\,M} = 0\,. \tag{6.11}
$$

As the result of gauge fixings (6.3), (6.9) and (6.11), the general action (4.15), the equations of motion and the target space gauge transformations are reduced to

$$
S_{q,\omega} = \int \mu^{-2,-2}\{q^{1,1\,M}D^{0,2}\omega^{1,-1\,M} + q^{1,1\,M}D^{2,0}\omega^{-1,1\,M} + H^{2,2}(q^{1,1},\omega^{1,-1},\omega^{-1,1},u,v)\}\,, \tag{6.12}
$$

$$
\begin{aligned}
D^{2,0}\omega^{-1,1\,M} + D^{0,2}\omega^{1,-1\,M} &= -\frac{\partial H^{2,2}(q,\omega,u,v)}{\partial q^{1,1\,M}}\,,\\
D^{2,0}q^{1,1\,M} = \frac{\partial H^{2,2}(q,\omega,u,v)}{\partial \omega^{-1,1\,M}}\,,\quad D^{0,2}q^{1,1\,M} &= \frac{\partial H^{2,2}(q,\omega,u,v)}{\partial \omega^{1,-1\,M}}\,. & (6.13)
\end{aligned}
$$

$$\delta H^{2,2} \;=\; \partial^{2,0}\Lambda^{0,2} + \partial^{0,2}\Lambda^{2,0} \tag{6.14}$$

$$\delta q^{1,1\,N} \;=\; \frac{\partial \Lambda^{2,0}}{\partial \omega^{1,-1\,N}}\;, \quad \delta \omega^{1,-1\,N} = -\frac{\partial \Lambda^{2,0}}{\partial q^{1,1\,N}}\;, \quad \delta \omega^{-1,1\,N} = -\frac{\partial \Lambda^{0,2}}{\partial q^{1,1\,N}}\;. \tag{6.15}$$

In (6.14), (6.15)

$$\begin{aligned} \Lambda^{2,0} &= \lambda^{2,0}(q,u,v) + \omega^{1,-1\,N}\lambda^{1,1\,N}(q,u,v)\;, \\ \Lambda^{0,2} &= \lambda^{0,2}(q,u,v) + \omega^{-1,1\,N}\lambda^{1,1\,N}(q,u,v)\;. \end{aligned} \tag{6.16}$$

We stress that on the way from the most general action (4.11) to the action (6.12) we did not make any extra assumptions: we only exploited the target space gauge freedom and some consequences of the general integrability condition (4.28). As we see, the target vielbeins $E^{-1,3\,N} \equiv D^{0,2}\omega^{-1,1\,N}$, $E^{3,-1\,N} \equiv D^{2,0}\omega^{1,-1\,N}$ entirely drop out from the dynamical equations (6.13) and so must be regarded as a sort of auxiliary, redundant quantities in the analytic target space geometry, in accord with the conjecture in the end of Sect.4 [4]. Nevertheless, one is still left with three equations for the four unknowns $E^{1,3\,N} \equiv D^{0,2}q^{1,1\,N}$, $E^{3,1\,N} \equiv D^{2,0}q^{1,1\,N}$, $E^{1,1\,N} \equiv D^{0,2}\omega^{1,-1\,N}$, $\tilde{E}^{1,1\,N} \equiv D^{2,0}\omega^{-1,1\,N}$. As we will see soon, the purely bosonic θ zero part of one of these vielbeins is actually a gauge degree of freedom due to the existence of gauge invariance generalizing the invariance (4.6) of the dual twisted multiplets action (4.1).

To reveal this invariance, one should further explore the consequences of the integrability condition (4.28) for the surviving potential $H^{2,2}$.

We proceed in the same way as in the $n = 1$ example. The $n \geq 2$ generalization of the conditions (5.14), (5.15) proves to be

$$\frac{\partial^2 H^{2,2}}{\partial \omega^{-1,1\,N}\partial \omega^{-1,1\,M}} \;=\; \frac{\partial^2 H^{2,2}}{\partial \omega^{1,-1\,N}\partial \omega^{1,-1\,M}} \;=\; \frac{\partial^2 H^{2,2}}{\partial \omega^{1,-1\,(N}\partial \omega^{-1,1\,M)}} \;=\; 0\;, \tag{6.17}$$

$$\begin{aligned} &\left(\partial^{2,0} + \frac{\partial H^{2,2}}{\partial \omega^{-1,1\,N}}\,\frac{\partial}{\partial q^{1,1\,N}} - \frac{1}{2}\,\frac{\partial H^{2,2}}{\partial q^{1,1\,N}}\,\frac{\partial}{\partial \omega^{-1,1\,N}}\right)\frac{\partial H^{2,2}}{\partial \omega^{1,-1\,M}} \\ &- \left(\partial^{0,2} + \frac{\partial H^{2,2}}{\partial \omega^{1,-1\,N}}\,\frac{\partial}{\partial q^{1,1\,N}} - \frac{1}{2}\,\frac{\partial H^{2,2}}{\partial q^{1,1\,N}}\,\frac{\partial}{\partial \omega^{1,-1\,N}}\right)\frac{\partial H^{2,2}}{\partial \omega^{-1,1\,M}} \;=\; 0 \end{aligned} \tag{6.18}$$

Eqs. (6.17) imply

$$\begin{aligned} H^{2,2} \;=\; & h^{2,2}(q,u,v) + \omega^{1,-1\,N}h^{1,3\,N}(q,u,v) + \omega^{-1,1\,N}h^{3,1\,N}(q,u,v) \\ & + \omega^{-1,1\,N}\omega^{1,-1\,M}h^{2,2\,[N,M]}(q,u,v)\;. \end{aligned} \tag{6.19}$$

Notice the presence of the term bilinear in ωs in the general case. Substituting this expression into eq. (6.18), we finally derive four constraints on the potentials $h^{2,2}$, $h^{1,3\,N}$, $h^{3,1\,N}$ and $h^{2,2\,[N,M]}$

$$\nabla^{2,0}h^{1,3\,N} - \nabla^{0,2}h^{3,1\,N} + h^{2,2\,[N,M]}\,\frac{\partial h^{2,2}}{\partial q^{1,1\,M}} \;=\; 0 \tag{6.20}$$

[4]One can view the conditions (4.26), (4.27) as the *definition* of $E^{-1,3\,N}$, $E^{3,-1\,N}$. These harmonic differential equations can be solved for $E^{-1,3\,N}$, $E^{3,-1\,N}$, thus expressing the latter through the remaining vielbeins (nonlocally in harmonics).

$$\nabla^{2,0}h^{2,2\,[N,M]} - \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,M]} + \frac{\partial h^{3,1\,M}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,N]} \;=\; 0 \tag{6.21}$$

$$\nabla^{0,2}h^{2,2\,[N,M]} - \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,M]} + \frac{\partial h^{1,3\,M}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,N]} \;=\; 0 \tag{6.22}$$

$$h^{2,2\,[N,T]}\,\frac{\partial h^{2,2\,[M,L]}}{\partial q^{1,1\,T}} + h^{2,2\,[L,T]}\,\frac{\partial h^{2,2\,[N,M]}}{\partial q^{1,1\,T}} + h^{2,2\,[M,T]}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,T}} \;=\; 0\,. \tag{6.23}$$

Here

$$\nabla^{2,0} = \partial^{2,0} + h^{3,1\,N}\frac{\partial}{\partial q^{1,1\,N}}\,,\quad \nabla^{0,2} = \partial^{0,2} + h^{1,3\,N}\frac{\partial}{\partial q^{1,1\,N}}\,. \tag{6.24}$$

For further reference, we rewrite the action and the equations of motion through the newly defined potentials

$$\begin{aligned} S_{q,\omega} \;=\; & \int \mu^{-2,-2}\{\, q^{1,1\,M}D^{0,2}\omega^{1,-1\,M} + q^{1,1\,M}D^{2,0}\omega^{-1,1\,M} + \omega^{1,-1\,M}h^{1,3\,M} \\ & +\omega^{-1,1\,M}h^{3,1\,M} + \omega^{-1,1\,M}\omega^{1,-1\,N}\;h^{2,2\,[M,N]} + h^{2,2}\;\} \end{aligned} \tag{6.25}$$

$$\begin{aligned} & \left(D^{2,0}\delta^{MN} + \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,M}}\right)\omega^{-1,1\,N} + \left(D^{0,2}\delta^{MN} + \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,M}}\right)\omega^{1,-1\,N} \\ & +\omega^{-1,1\,S}\omega^{1,-1\,T}\;\frac{\partial h^{2,2\,[S,T]}}{\partial q^{1,1\,M}} \;=\; -\frac{\partial h^{2,2}}{\partial q^{1,1\,M}}\,, \\ & D^{2,0}q^{1,1\,M} - h^{3,1\,M} + \omega^{1,-1\,N}h^{2,2\,[N,M]} \;=\; 0 \\ & D^{0,2}q^{1,1\,M} - h^{1,3\,M} - \omega^{-1,1\,N}h^{2,2\,[N,M]} \;=\; 0 \end{aligned} \tag{6.26}$$

These action and equations enjoy a rich set of invariances.

One of them is the form-invariance under the restricted target space reparametrizations (6.14), (6.15). They are realized on the superfields and potentials in the following way

$$\begin{aligned} \delta q^{1,1\,N} \;&=\; \lambda^{1,1\,N}\,,\quad \delta\omega^{-1,1\,N} \;=\; -\frac{\partial\lambda^{0,2}}{\partial q^{1,1\,N}} - \frac{\partial\lambda^{1,1\,M}}{\partial q^{1,1\,N}}\;\omega^{-1,1\,M}\,, \\ \delta\omega^{1,-1\,N} \;&=\; -\frac{\partial\lambda^{2,0}}{\partial q^{1,1\,N}} - \frac{\partial\lambda^{1,1\,M}}{\partial q^{1,1\,N}}\;\omega^{1,-1\,M}\,, \\ \delta h^{2,2} \;&=\; \nabla^{2,0}\lambda^{0,2} + \nabla^{0,2}\lambda^{2,0}\,, \\ \delta h^{3,1\,M} \;&=\; \nabla^{2,0}\lambda^{1,1\,M} + h^{2,2\,[M,N]}\;\frac{\partial\lambda^{2,0}}{\partial q^{1,1\,N}} \\ \delta h^{1,3\,M} \;&=\; \nabla^{0,2}\lambda^{1,1\,M} - h^{2,2\,[M,N]}\;\frac{\partial\lambda^{0,2}}{\partial q^{1,1\,N}} \\ \delta h^{2,2\,[N,M]} \;&=\; \frac{\partial\lambda^{1,1\,N}}{\partial q^{1,1\,L}}\;h^{2,2\,[L,M]} - \frac{\partial\lambda^{1,1\,M}}{\partial q^{1,1\,L}}\;h^{2,2\,[L,N]}\,. \end{aligned} \tag{6.27}$$

It is a simple exercise to directly check the covariance of constraints (6.20) - (6.23) under these reparametrizations.

More interesting is the gauge invariance inherent to the action (6.25). It is a highly nontrivial nonabelian (and in general nonlinear) generalization of the gauge invariance (4.6)

$$
\begin{aligned}
\delta\omega^{1,-1\,M} &= \left(D^{2,0}\delta^{MN} + \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,M}}\right)\sigma^{-1,-1\,N} - \omega^{1,-1\,L}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,M}}\,\sigma^{-1,-1\,N}\,, \\
\delta\omega^{-1,1\,M} &= -\left(D^{0,2}\delta^{MN} + \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,M}}\right)\sigma^{-1,-1\,N} - \omega^{-1,1\,L}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,M}}\,\sigma^{-1,-1\,N}\,, \\
\delta q^{1,1\,M} &= \sigma^{-1,-1\,N} h^{2,2\,[N,M]}\,.
\end{aligned} \tag{6.28}
$$

As expected, the action is invariant only provided the integrability conditions (6.20) - (6.23) are obeyed. In general, these gauge transformations close with a field-dependent Lie bracket parameter. Commuting two such transformations on $q^{1,1\,N}$, and using the cyclic constraint (6.23), we find

$$
\delta_{br} q^{1,1\,M} = \sigma_{br}^{-1,-1\,N} h^{2,2\,[N,M]}\,, \quad \sigma_{br}^{-1,-1\,N} = -\sigma_1^{-1,-1\,L}\sigma_2^{-1,-1\,T}\frac{\partial h^{2,2\,[L,T]}}{\partial q^{1,1\,N}}\,. \tag{6.29}
$$

We see that eq. (6.23) ensures the nonlinear closure of the algebra of gauge transformations (6.28) and so it is a group condition similar to the Jacobi identity. It is curious that the gauge transformations (6.28) with the relation (6.23) are precise bi-harmonic counterparts of the basic relations of a two-dimensional version of the recently proposed nonlinear extension of Yang-Mills theory, so called “Poisson gauge theory” [20] (with the evident correspondence $D^{2,0}, D^{0,2} \leftrightarrow \partial_\mu$; $\omega^{1,-1\,M}, -\omega^{-1,1\,M} \leftrightarrow A^M_\mu$).

We point out that it is the presence of the antisymmetric potential $h^{2,2\,[N,M]}$ that makes the considered case nontrivial and, in particular, the gauge invariance (6.28) nonabelian. If $h^{2,2\,[N,M]}$ is vanishing, the invariance gets abelian and the constraints (6.20) - (6.23) except for (6.20) are identically satisfied, while (6.20) can be solved on the pattern of the $n = 1$ case, eqs. (5.20). As a result, the potentials $h^{1,3\,N}$, $h^{3,1\,N}$ can be gauged away using the $\lambda^{1,1\,N}$ freedom (6.27), and we return to the general twisted multiplet action (4.1). On the contrary, with nonvanishing $h^{2,2\,[N,M]}$ eq. (6.20) does not imply $h^{1,3\,N}$, $h^{3,1\,N}$ to be pure gauge. We cannot remove the ω dependence from second and third of eqs. (6.26) by any local field redefinition with preserving harmonic analyticity. Moreover, in contradistinction to the constraints (3.7), these equations are compatible only with using the first equation. So, the obtained system definitely does not admit in general an equivalent description in terms of twisted $(4,4)$ analytic superfields. Hence, the left and right complex structures on the target space can be non-commuting and we will see soon that this is indeed so for non-vanishing $h^{2,2\,[N,M]}$. On the other hand, $q^{1,1\,N}$ can be expressed by first of eqs. (6.26) (at least, iteratively) via ω superfields to yield the ω representation of the action similar to (4.4). The main distinguishing feature of this general ω action is the nonlinear and nonabelian gauge symmetry.

To avoid a misunderstanding, we note that the analogies with two-dimensional gauge theories are somewhat formal because there is no any genuine propagating gauge field among the components of the superfields ω. The only practical role of the gauge freedom (6.28) seems to consist in ensuring the correct number of physical degrees of freedom

in the action (6.25) (after elimination of $q^{1,1\,N}$ by its algebraic equation of motion). It is also unclear, in what sense the transformations (4.6), (6.28) could be interpreted as gauging of some rigid ones. Indeed, in the present case the naive definition of the rigid group via imposing the conditions $D^{2,0}\sigma^{-1,-1\,M} = D^{0,2}\sigma^{-1,-1\,M} = 0$ leads to the trivial result $\sigma^{-1,-1\,M} = 0$. Nevertheless, this gauge symmetry is a necessary ingredient of the manifestly supersymmetric off-shell unconstrained superfield description of torsionful $(4,4)$ sigma models. It should be taken into account, e.g., while quantizing these models in the harmonic superfield formalism (one is led to introduce the appropriate Faddeev-Popov ghosts, etc). It certainly plays an important role in the analytic bosonic target space geometry of the models in question. Indeed, by analogy with the hyper-Kähler case [7], the basic relations of this geometry are expected to be the θ independent parts of the superfield equations of motion (6.13) (or their more detailed form (6.26)). They relate the target space harmonic vielbeins $E^{1,3\,N}$, $E^{3,1\,N}$, $E^{1,1\,N}$ and $\tilde{E}^{1,1\,N}$ to the potential $H^{2,2}$. The gauge invariance we are discussing allows us to completely gauge away one of the vielbeins $E^{1,1\,N}$, $\tilde{E}^{1,1\,N}$ (by gauging away either $\omega^{1,-1\,N}|_{\theta=0}$ or $\omega^{-1,1\,N}|_{\theta=0}$) [5] and, thereby, to match the number of vielbeins with that of independent equations. Of course, the group-theoretical and geometric meaning of this important gauge freedom still needs to be clarified.

It remains to solve the constraints (6.20) - (6.23). They have a nice geometric form and certainly encode a nontrivial geometry. For the time being, we are not aware of their general solution and are able to present only a particular one. Nonetheless, it is rather interesting on its own and seems to share most of characteristic features of the general case.

7 Harmonic Yang-Mills sigma models

The particular solution we just mentioned is given by the following ansatz

$$\begin{aligned} h^{1,3\,N} &= h^{3,1\,N} = 0\,;\; h^{2,2} = h^{2,2}(t,u,v)\,,\;\; t^{2,2} = q^{1,1\,M}q^{1,1\,M}\,; \\ h^{2,2\,[N,M]} &= b^{1,1} f^{NML} q^{1,1\,L}\,,\; b^{1,1} = b^{ia}u^1_i v^1_a\,,\; b^{ia} = \text{const}\,, \end{aligned} \tag{7.1}$$

where the constants f^{NML} are real and totally antisymmetric. The constraints (6.20) - (6.22) are identically satisfied with this ansatz, while (6.23) is now none other than the Jacobi identity which implies the constants f^{NML} to be the structure constants of some real semi-simple Lie algebra (the minimal possibility is $n = 3$, the corresponding algebra being $so(3)$). Thus the $(4,4)$ sigma models associated with the ansatz (7.1) are to be treated as a sort of Yang-Mills theories in the $SU(2)\times SU(2)$ harmonic superspace. They give a natural nonabelian generalization of the twisted multiplet sigma models with the action (4.1) which, as was noticed in Sect.4, are analogs of two-dimensional abelian gauge theory. The action (6.25), equations of motion (6.26) and the gauge transformation laws (6.28) specialized to the case (7.1) read

$$\begin{aligned} S^{YM}_{q,\omega} = \int \mu^{-2,-2}\{\, & q^{1,1\,M}(\, D^{0,2}\omega^{1,-1\,M} + D^{2,0}\omega^{-1,1\,M} + b^{1,1}\,\omega^{-1,1\,L}\omega^{1,-1\,N}f^{LNM}\,) \\ & + h^{2,2}(q,u,v)\} \end{aligned} \tag{7.2}$$

[5] This is not the case at the full superfield level, see eq. (7.7).

$$
\begin{aligned}
&D^{2,0}\omega^{-1,1\,N} + D^{0,2}\omega^{1,-1\,N} + b^{1,1}\,\omega^{-1,1\,S}\omega^{1,-1\,T}f^{STN} \;\equiv\; B^{1,1\,N} \;=\; -\frac{\partial h^{2,2}}{\partial q^{1,1\,N}}\,,\\
&D^{2,0}q^{1,1\,M} + b^{1,1}\,\omega^{1,-1\,N}f^{NML}q^{1,1\,L} \;\equiv\; \Delta^{2,0}q^{1,1\,M} \;=\; 0\\
&D^{0,2}q^{1,1\,M} - b^{1,1}\,\omega^{-1,1\,N}f^{NML}q^{1,1\,L} \;\equiv\; \Delta^{0,2}q^{1,1\,M} \;=\; 0 \qquad (7.3)\\
&\delta\omega^{1,-1\,M} \;=\; \Delta^{2,0}\sigma^{-1,-1\,M}\,,\ \delta\omega^{-1,1\,M} \;=\; -\Delta^{0,2}\sigma^{-1,-1\,M}\,,\\
&\delta q^{1,1\,M} \;=\; b^{1,1}\,\sigma^{-1,-1\,N}f^{NML}q^{1,1\,L}\,. \qquad (7.4)
\end{aligned}
$$

Now the analogy with two-dimensional nonabelian gauge theory becomes almost literal, especially for

$$h^{2,2} = q^{1,1\,M}q^{1,1\,M}\,. \qquad (7.5)$$

With this choice,

$$q^{1,1\,N} = -\frac{1}{2}\,B^{1,1\,N}$$

in virtue of the first of eqs. (7.3), then two remaining equations are direct analogs of two-dimensional Yang-Mills equations

$$\Delta^{2,0}B^{1,1\,N} = \Delta^{0,2}B^{1,1\,N} = 0\,, \qquad (7.6)$$

and we recognize (7.2) and (7.3) as the harmonic counterpart of the first order formalism of two-dimensional Yang-Mills theory. In the general case $q^{1,1\,M}$ is a nonlinear function of $B^{1,1\,N}$, however $B^{1,1\,N}$ still obeys the same equations (7.6).

It is instructive to see how the fundamental integrability condition (4.8) is satisfied with the ansatz (7.1):

$$[\Delta^{2,0},\Delta^{0,2}]\,q^{1,1\,M} = -b^{1,1}\,B^{1,1\,N}f^{NML}q^{1,1\,L} = 2b^{1,1}\,\frac{\partial h^{2,2}}{\partial t^{2,2}}\,q^{1,1\,N}f^{NML}q^{1,1\,L} \equiv 0\,.$$

We stress once more that in checking this condition in the nonabelian case one necessarily needs first of eqs. (7.3), while in the abelian, twisted multiplet case (4.1) the integrability condition is satisfied without any help from eq. (4.2). In other words, in the nonabelian case we cannot interpret the ω equations of motion as some independent kinematical constraints on $q^{1,1\,N}$: they are self-consistent only together with the q equation. As was already mentioned, this property reflects the fact that the class of $(4,4)$ sigma models we have found cannot be described only in terms of twisted $(4,4)$ multiplets (of course, in general the above gauge group has the structure of a direct product which can include abelian factors; the relevant $q^{1,1}$'s satisfy the linear twisted multiplet constraints (3.11)).

An interesting feature of this "harmonic Yang-Mills theory" is the presence of the doubly charged "coupling constant" $b^{1,1} = b^{ia}u^1_i v^1_a$, which is necessary for the correct balance of the harmonic $U(1)$ charges. Thus in the geometry of the considered class of $(4,4)$ sigma models an essential role belongs to some quartet constant b^{ia}. In the limit $b^{ia} \to 0$ the nonabelian structure contracts into the abelian one and we reproduce the twisted multiplet action (4.1). As we will see soon, this constant measures the "strength" of non-commutativity of the left and right quaternionic structures on the target space: in the contraction limit these structures become mutually commuting.

In forthcoming publications we will present more detailed study of all these issues, including those related to the target space geometry and complex structures, at the component level, both for the general case and the Yang-Mills example at hand. In the rest of this Section we give, to the first non-vanishing order in physical bosonic fields, the bosonic metric and torsion potential, as well as the left and right complex structures for the Yang-Mills ansatz (7.1). Our purpose will be to explicitly demonstrate the non-commutativity of complex structures for $b^{ia} \neq 0$ in (7.1). For simplicity we take $h^{2,2}$ in the form (7.5).

As a first step we impose the following Wess-Zumino gauge with respect to the local symmetry (7.4)

$$\omega^{1,-1\,N}(\zeta,u,v) = \theta^{1,0\,\underline{i}}\,\nu^{0,-1\,N}_{\underline{i}}(\zeta_R,v) + \theta^{1,0}\theta^{1,0}\;g^{0,-1\,iN}(\zeta_R,v)u^{-1}_i \tag{7.7}$$

with

$$\{\zeta_R\} \equiv \{x^{++}, x^{--}, \theta^{0,1\,\underline{a}}\}\;.$$

Note that with this choice there remains no any residual gauge invariance, though all the relations below still respect a rigid invariance under the transformations of the group with structure constants f^{MNL} (it acts as some rotations in indices $M, N, ...$). Then we substitute (7.7) into (7.2) with $h^{2,2}$ given by (7.5), integrate over θ's and u's, eliminate infinite tails of decoupling auxiliary fields and, finally, obtain the physical bosons part of the action (7.2) as the following integral over x and harmonics v

$$S_{bos} = \int d^2x[dv]\left(\frac{i}{2}\,g^{0,-1\,iM}(x,v)\,\partial_{--}q^{0,1\,M}_i(x,v)\right)\;. \tag{7.8}$$

Here the fields g and q satisfy the harmonic differential equations

$$\begin{aligned} \partial^{0,2}g^{0,-1\,iM} - 2(b^{ka}v^1_a)\;f^{MNL}q^{0,1\,iN}g^{0,-1\,L}_k \;&=\; 4i\partial_{++}q^{0,1\,iM} \\ \partial^{0,2}q^{0,1\,iM} - 2f^{MLN}(b^{ka}v^1_a)\;q^{0,1\,L}_k\;q^{0,1\,iN} \;&=\; 0\,. \end{aligned} \tag{7.9}$$

They are related to the initial superfields as

$$q^{1,1\,M}(\zeta,u,v)| = q^{0,1\,iM}(x,v)u^1_i + ...\;,\quad g^{0,-1\,iN}(\zeta_R,v)| = g^{0,-1\,iN}(x,v)\;,$$

where $|$ means restriction to the θ independent parts.

In order to represent the action as an integral over x^{++}, x^{--}, we should solve eqs. (7.9), substitute the solution into (7.8) and perform the v integration. Here we limit ourselves to solving (7.9) to the first non-vanishing order in the physical bosonic field $q^{ia\,M}(x)$, the first component in the v expansion of $q^{0,1\,iM}$

$$q^{0,1\,iM}(x,v) = q^{ia\,M}(x)v^1_a + ...\;.$$

Representing (7.8) as

$$S_{bos} = \int d^2x\left(G^{M\;L}_{ia\;kb}\partial_{++}q^{ia\,M}\partial_{--}q^{kb\,L} + B^{M\;L}_{ia\;kb}\partial_{++}q^{ia\,M}\partial_{--}q^{kb\,L}\right) \tag{7.10}$$

where the metric G and the torsion potential B are, respectively, symmetric and skew-symmetric with respect to the simultaneous permutation of the left and right sets of their indices, we find

$$G^{M\;L}_{ia\;kb} = \delta^{ML}\epsilon_{ik}\epsilon_{ab} - \frac{2}{3}\epsilon_{ik}f^{MLN}b_{l(a}q^{l\,N}_{b)} + ...\;,\;B^{M\;L}_{ia\;kb} = \frac{2}{3}f^{MLN}[b_{(ia}q^N_{k)b} + b_{(ib}q^N_{k)a}] + ...\;. \tag{7.11}$$

Note that an asymmetry between the indices ik and ab in the metric is related to our choice of the WZ gauge (7.7). One could choose another gauge to restore a symmetry between the above pairs of $SU(2)$ indices. Metrics in different gauges are connected via the target space $q^{ia\,M}$ reparametrizations.

Finally, let us compute, again to the first order in $q^{ia\,M}$, the relevant left and right complex structures. According to the well-known strategy [2, 3, 18], we need: (i) to partially go on shell by eliminating the auxiliary fermionic fields; (ii) to divide four supersymmetries in every light-cone sector into a $N = 1$ one which is realized linearly and a triplet of nonlinearly realized extra supersymmetries; (iii) to consider the transformation laws of the physical bosonic fields $q^{ia\,M}$ under extra supersymmetries. The complex structures can be read off from these transformation laws.

In our case at the step (i) we solve some harmonic differential equations of motion in order to express an infinite tail of auxiliary fermionic fields in terms of the physical ones and the bosonic fields $q^{ia\,M}$. At the step (ii) we single out the $(1,1)$ supersymmetry by decomposing the $(4,0)$ and $(0,4)$ supersymmetry parameters $\varepsilon_-^{i\underline{i}}$ and $\varepsilon_+^{a\underline{a}}$ as

$$\varepsilon^{i\underline{i}\,+} \equiv \epsilon^{i\underline{i}}\varepsilon^+ + i\varepsilon^{(i\underline{i})\,+}\ ,\quad \varepsilon^{a\underline{a}\,-} \equiv \epsilon^{a\underline{a}}\varepsilon^- + i\varepsilon^{(a\underline{a})\,-}\ ,$$

where we have kept a manifest symmetry only with respect to the diagonal $SU(2)$ groups in the full left and right automorphism groups $SO(4)_L$ and $SO(4)_R$. At the step (iii) we redefine the physical fermionic fields so that the singlet supersymmetries with the parameters ε_- and ε_+ are realized linearly. We skip the details and present the final form of the on-shell supersymmetry transformations of $q^{ia\,M}(x)$

$$\delta q^{ia\,M} = \varepsilon^+\ \psi_+^{ia\,M} + i\varepsilon^{(kj)\,+}\ \left(F_{(kj)}\right)^{ia\,M}_{lb\,L}\ \psi_+^{lb\,L} + \varepsilon^-\ \chi_-^{ia\,M} + i\varepsilon^{(cd)\,-}\ \left(\hat{F}_{(cd)}\right)^{ia\,M}_{lb\,L}\ \chi_-^{lb\,L}\ . \quad (7.12)$$

Introducing the matrices

$$F^n_{(+)} \equiv (\tau^n)^k_j F^{(j}{}_{k)}\ ,\quad F^m_{(-)} \equiv (\tau^m)^c_d \hat{F}^{(d}{}_{c)}\ ,$$

τ^n being Pauli matrices, we find that in the first order in $q^{ia\,M}$ and b^{ia}

$$\begin{aligned} F^n_{(+)} &= -i\tau^n \otimes I \otimes I + \frac{i}{3}\,[\,M_{(+)}, \tau^n \otimes I \otimes I\,] \\ F^n_{(-)} &= -iI \otimes \tau^n \otimes I + \frac{i}{3}\,[\,M_{(-)}, I \otimes \tau^n \otimes I\,] \end{aligned} \quad (7.13)$$

$$\left(M_{(+)}\right)^{ia\,M}_{kb\,N} = -2\ f^{MLN}\left(b_b^{(i}q^{a\,L}{}_{k)} + b^{(ia}q^L{}_{k)b}\right)\ ,\quad \left(M_{(-)}\right)^{ia\,M}_{kb\,N} = 2\ f^{MLN}\ b^i_{(b}q^{a)\,L}_k\ , \quad (7.14)$$

where the matrix factors in the tensor products are arranged so that they act, respectively, on the indices $i, j, k, ..., a, b, c, ..., M, N, L, ...$.

It is easy to see that the matrices $F^n_{(\pm)}$ to the first order in q, b possess all the standard properties of complex structures needed for on-shell $(4,4)$ supersymmetry [2, 3]. In particular, they form a quaternionic algebra

$$F^n_{(\pm)}F^m_{(\pm)} = -\delta^{nm} + \epsilon^{nms}F^s_{(\pm)}\ ,$$

and satisfy the covariant constancy conditions

$$\mathcal{D}_{lc\,K}\left(F^{n}_{(\pm)}\right)^{ia\,M}_{kb\,N} = \partial_{lc\,K}\left(F^{n}_{(\pm)}\right)^{ia\,M}_{kb\,N} - \Gamma_{(\pm)\,lc\,K\;\;kb\,N}^{\;\;\;\;\;\;\;\;\;jd\,T}\left(F^{n}_{(\pm)}\right)^{ia\,M}_{jd\,T} + \Gamma_{(\pm)\,lc\,K\;\;jd\,T}^{\;\;\;\;\;\;\;\;\;ia\,M}\left(F^{n}_{(\pm)}\right)^{jd\,T}_{kb\,N} = 0$$

with

$$\Gamma_{(\pm)\,lc\,M\;\;kb\,N}^{\;\;\;\;\;\;\;\;\;jd\,T} \equiv \Gamma_{lc\,M\;\;kb\,N}^{\;\;\;\;\;\;jd\,T} \mp T_{lc\,M\;\;kb\,N}^{\;\;\;\;\;\;jd\,T}\,,$$

where Γ is the standard Riemann connection for the metric (7.11) and T is the torsion

$$T_{ia\,M\;\;kb\,N\;\;ld\,T} = \frac{1}{2}\left(\partial_{ia\,M}B^{N\,T}_{kb\,ld} + \;\; cyclic\right)\;.$$

It is also straightforward to check two remaining conditions of the on-shell $(4,4)$ supersymmetry [2,3]. In the present case all these requirements are guaranteed to be automatically fulfilled because we proceeded from a manifestly $(4,4)$ supersymmetric off-shell superfield formulation.

It remains to compute the commutator of complex structures. We find (again, to the first order in fields)

$$\begin{aligned}
[\,F^{n}_{(+)}, F^{m}_{(-)}\,] \;&=\; (\tau^n\otimes I\otimes I)M_{(-)}(I\otimes\tau^m\otimes I) + (I\otimes\tau^m\otimes I)M_{(-)}(\tau^n\otimes I\otimes I) \\
&-\; (\tau^n\otimes\tau^m\otimes I)M_{(-)} - M_{(-)}(\tau^n\otimes\tau^m\otimes I)\;\neq\;0\,. \qquad (7.15)
\end{aligned}$$

Thus in the present case in the bosonic sector we encounter a more general geometry compared to the one associated with twisted $(4,4)$ multiplets. The basic characteristic feature of this geometry is the non-commutativity of the left and right complex structures. It is easy to check this property also for general potentials $h^{2,2}(q,u,v)$ in (7.2)[6]. It seems obvious that the general case (6.25), (6.20) - (6.23) reveals the same feature. Stress once more that this important property is related in a puzzling way to the nonabelian structure of the analytic superspace actions (7.2), (6.25): the coupling constant $b^{1,1}$ (or the Poisson potential $h^{2,2\,[M,N]}$ in the general case) measures the strength of the non-commutativity of complex structures.

The main purpose of this Section was to explicitly show that in the $(4,4)$ models we have constructed the left and right complex structures on the bosonic target do not commute. For full understanding of the geometry of these models, at least in the particular case discussed in this Section, and for clarifying its relation to the known examples, e.g., to the group manifold ones [21], we need the explicit form of the metrics and torsion potentials in (7.10). This amounts to finding the complete (non-iterative) solution to eqs. (7.9) and their generalization to the case of non-trivial potentials $h^{2,2}(t,u,v)$ in (7.2). A work along this line is now in progress. We wish to point out that one of the merits of the off-shell formulation proposed consists in the fact that, similarly to the case of hyper-Kähler $(4,4)$ sigma models [8] or $(4,0)$ models [12], we can *explicitly* compute the bosonic metrics starting from the unconstrained superfield action (7.2) (or its generalization corresponding to the general solution of constraints (6.20) - (6.23)). These metrics are guaranteed to satisfy all the conditions of on-shell $(4,4)$ supersymmetry.

[6] This means, in particular, that a subclass of metrics associated with twisted $(4,4)$ multiplets, for dimensions $4n,\ n\geq 3$, admits a deformation which preserves $(4,4)$ SUSY but makes the left and right complex structures non-commuting.

Though we are not yet aware of the detailed properties of the corresponding bosonic metrics (singularities, etc.), in the particular case (7.1) we know some of their isometries. Namely, as was already mentioned, the action (7.2) and its bosonic part (7.10) (for any choice of $h^{2,2}(t,u,v)$ in (7.1)) respect invariance under the global transformations of the group with structure constants f^{MNL}. This suggests a link with the group manifold $(4,4)$ models [21].

Our last comment concerns the relation with the recent paper by Delduc and Sokatchev [18]. They studied a superfield description of $(2,2)$ sigma models with non-commuting structures and found a set of nonlinear constraints on the Lagrangian which somewhat resemble eqs. (6.20) - (6.23). An essential difference of their approach from ours seems to lie in that it does not allow a smooth limiting transition to the case with commuting structures.

8 Summary and outlook

For reader's convenience, we summarize here the basic steps and results of our analysis.

We proceeded from the dual action (4.1) of $(4,4)$ twisted multiplet in the analytic harmonic $SU(2) \times SU(2)$ superspace and wrote down its most general conceivable extension (4.11) involving n copies of the triple of analytic harmonic superfields $q^{1,1\,M}$, $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ ($M = 1, ...n$). Then, using a freedom with respect to the redefinitions (4.12) and (4.13), we reduced it to the form (4.15). It has been further simplified, to the form (6.12), by using the residual target space gauge freedom (4.16), (4.17) together with some consequences of the integrability condition (4.28) which stems from the commutativity of the harmonic derivatives $D^{2,0}$ and $D^{0,2}$. After this we studied further restrictions imposed on the structure of the action (6.12) by the integrability condition (4.28). The latter entirely fixes the ω dependence of the superfield Lagrangian, bringing the action to the form (6.25) with the potentials $h^{2,2}$, $h^{1,3\,N}$, $h^{3,1\,N}$ and $h^{2,2\,[N,M]}$ constrained by eqs. (6.20) - (6.23). The action (6.25) reveals new features compared to the twisted multiplet action (4.1) only provided the potential $h^{2,2\,[N,M]}$ is non-vanishing; otherwise, (6.25) can be reduced to (4.1) by a field redefinition. For $n = 1$ the potential $h^{2,2\,[N,M]}$ identically vanishes, so the novel class of $(4,4)$ sigma model actions with non-zero $h^{2,2\,[N,M]}$ exists beginning with $n = 2$. Its main novelty is the nonabelian and in general nonlinear gauge invariance (6.28) which substitutes the abelian gauge invariance (4.6) of the twisted multiplets action. These new actions involve an infinite number of auxiliary fields and do not admit a formulation in terms of the twisted $(4,4)$ superfields only. They provide an off-shell description of $(4,4)$ sigma models with non-commuting left and right triplets of complex structures.

There remains a lot of things to be done and questions to be answered. Besides a general problem of inquiring the intrinsic geometric aspects of the action (6.25) and constraints (6.20) - (6.23) as well as revealing their links with the full target space geometry, there are a few more specific (and urgent) ones two of which we will mention here.

An interesting problem is to examine whether the constraints (6.20) - (6.23) admit solutions corresponding to $(4,4)$ supersymmetric WZNW sigma models on the group manifolds from the list given in [21]. Only for the simplest manifolds from this list, namely $[U(1)]^4$ and $SU(2) \times U(1)$, the left and right complex structures commute [16] and only

for the related WZNW models there exists a description via twisted multiplets (in the $q^{1,1}$ language, these models are described by the free action (3.9) and the action (3.12), respectively). On higher-dimensional manifolds which are not reduced to products of these two, the left and right structures do not commute. We conjecture that the associated $(4,4)$ WZNW sigma models are described off shell by the actions (6.25) with proper potentials $h^{2,2\,[N,M]}$. The minimal number of the superfield triples at which $h^{2,2\,[N,M]}$ exists, $n = 2$, amounts to the dimension 8 of the bosonic target. This precisely matches with the dimension of the first nontrivial manifold from the aforementioned list, that of the group $SU(3)$.

One more problem is to prove that the general action of the triples $q^{1,1}, \omega^{1,-1}, \omega^{-1,1}$ in the analytic $SU(2) \times SU(2)$ harmonic superspace indeed yields a most general $(4,4)$ supersymmetric sigma model with torsion. Our starting point in this paper was the analytic superfield $q^{1,1}$ which represents a $(4,4)$ twisted multiplet. But this is merely one type of $(4,4)$ twisted multiplet. There exist other types which display the same irreducible $(8+8)$ off-shell content, but differ in the $SU(2)_L \times SU(2)_R$ assignment of component fields (see, e.g., [22, 23]). For the time being it is unclear how to simultaneously decribe all these types within the same $SU(2) \times SU(2)$ harmonic superspace. Perhaps, they can be related to each other by a duality transformation (like all $N = 2$ $4D$ matter multiplets are related to the ultimate analytic $q^{(+)}$ multiplet [5]). Alternatively, it may happen that for their consistent description one will need to harmonize the whole $(4,4)$ supersymmetry automorphism group $SO(4)_L \times SO(4)_R$, i.e. to introduce two extra sets of $SU(2)$ harmonic variables, and to consider appropriate analytic superfields in this maximally extended $(4,4)$ harmonic superspace. The relevant actions will be certainly more general than those discussed in this paper. Clearly, in order to distinguish between all these possibilities, one should understand in full the geometry of the target space and various harmonic extensions of the latter for general $(4,4)$ sigma models with torsion, like this has been done for their hyper-Kähler counterparts in [7] and for $(4,0)$ sigma models in [11].

Finally, it would be interesting to find out possible implications of the considered class of $(4,4)$ sigma models in the superstring theories for which these sigma models could provide some consistent backgrounds.

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